

# LSZ in LST

*Ofer Aharony<sup>a</sup>, Amit Giveon<sup>b</sup> and David Kutasov<sup>c</sup>*

<sup>a</sup>Department of Particle Physics, Weizmann Institute of Science, Rehovot 76100, Israel

`Ofer.Aharony@weizmann.ac.il`

<sup>b</sup>Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

`giveon@vms.huji.ac.il`

<sup>c</sup>EFI and Department of Physics, University of Chicago

5640 S. Ellis Av., Chicago, IL 60637, USA

`kutasov@theory.uchicago.edu`

We discuss the analytic structure of off-shell correlation functions in Little String Theories (LSTs) using their description as asymptotically linear dilaton backgrounds of string theory. We focus on specific points in the LST moduli space where this description involves the spacetime  $\mathbb{R}^{d-1,1} \times SL(2)/U(1)$  times a compact CFT, though we expect our qualitative results to be much more general. We show that  $n$ -point functions of vertex operators  $\mathcal{O}(p_\mu)$  have single poles as a function of the  $d$ -dimensional momentum  $p_\mu$ , which correspond to normalizable states localized near the tip of the  $SL(2)/U(1)$  cigar. Additional poles arise due to the non-trivial dynamics in the bulk of the cigar, and these can lead to a type of UV/IR mixing. Our results explain some previously puzzling features of the low energy behavior of the Green functions. As another application, we compute the precise combinations of single-trace and multi-trace operators in the low-energy gauge theory which map to single string vertex operators in the  $\mathcal{N} = (1, 1)$  supersymmetric  $d = 6$  LST. We also discuss the implications of our results for two dimensional string theories and for the (non-existence of a) Hagedorn phase transition in LSTs.

## 1. Introduction and Summary

Little String Theory (for reviews see [1,2] and section 4 of [3]) describes the physics of defects, such as  $NS5$ -branes and/or singularities, in string theory. It can be isolated from the rest of the dynamics by taking the decoupling limit in which the string coupling far from the defect goes to zero [4,5]. The effective coupling near the defect remains non-vanishing and grows as one approaches the defect.

A number of methods to study the dynamics of Little String Theory (LST) have been proposed. One uses discrete light-cone quantization [6,7,8], in the spirit of Matrix theory. Another, which we will focus on here, is provided by string propagation in the near-horizon geometry of the defect [9], as in the AdS/CFT correspondence. LSTs are holographically equivalent to asymptotically linear dilaton backgrounds (which are sometimes called “non-critical string theories”). In the near horizon geometry, the radial direction away from the defect is described by a non-compact scalar  $\phi$ . For large positive  $\phi$ , the dilaton  $\Phi$  depends on  $\phi$  as follows:

$$g_s = e^\Phi \simeq e^{-\frac{Q}{2}\phi}, \quad (1.1)$$

where the slope of the linear dilaton,  $Q$ , is real and positive. The worldsheet central charge of  $\phi$  is given by

$$c_\phi = 1 + 3Q^2. \quad (1.2)$$

As  $\phi \rightarrow \infty$ , the string coupling (1.1) goes to zero and interactions turn off. Thus,  $\phi = \infty$  can be thought of as the boundary of the near-horizon geometry of the defect. As  $\phi$  decreases, the string coupling (1.1) grows, and there are two basic possibilities. If the string coupling remains small everywhere in the near-horizon spacetime, we can study the system using perturbative string techniques. If, on the other hand, the string coupling becomes of order one or larger anywhere, the system may not be perturbative, and one may have to resort to other means of studying it. In this paper, we will restrict the discussion to the weakly coupled case.

The full geometry around a defect corresponding to a  $d$ -dimensional LST has the typical form (near the boundary at  $\phi = \infty$ )

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{M} \quad (1.3)$$

where  $\mathbb{R}^{d-1,1}$  labels the worldvolume of the defect,  $\mathbb{R}_\phi$  is the real line labeled by  $\phi$ , and  $\mathcal{M}$  labels the angular directions at fixed distance from the defect. For example, for  $k$

parallel  $NS5$ -branes in type II string theory one has [10]  $d = 6$  and  $\mathcal{M} = SU(2)_k$ , the supersymmetric level  $k$   $SU(2)$  WZW model. At finite  $\phi$ , the geometry (1.3) must be deformed to avoid the strong coupling singularity at  $\phi = -\infty$ .

The spectrum of normalizable states in weakly coupled asymptotically linear dilaton spacetimes falls into two classes. One consists of delta-function normalizable states whose vertex operators behave at large  $\phi$  as  $\exp(-\frac{Q}{2} + i\lambda)\phi$  with real  $\lambda$ . These scattering states correspond to incoming and outgoing waves carrying momentum  $\lambda$  in the (radial)  $\phi$  direction. They are quite analogous to standard scattering states in critical string theory. From the point of view of the  $d$  dimensional theory on  $\mathbb{R}^{d-1,1}$ , they have a continuous mass spectrum, which generally starts above some mass gap (which depends on the theory and on the operator).

In addition to the delta-function normalizable scattering states, the theory typically includes normalizable states which live at finite  $\phi$ . The spectrum of such states is discrete; they can be thought of as bound states associated with the defect. They are described by normalizable vertex operators, with wavefunctions that decay rapidly as  $\phi \rightarrow \infty$ . For example, in the  $d = 6$ ,  $\mathcal{N} = (1, 1)$  LST corresponding to parallel  $NS5$ -branes in type IIB string theory that we will study in detail below, these states include the massless gauge bosons living on the fivebranes, and their superpartners.

In critical string theory, the physical observables are vertex operators corresponding to on-shell states. Their correlation functions give the S-matrix elements of these states. Such observables exist in the asymptotically linear dilaton (non-critical) case (1.3) as well. Indeed, one can use the (delta-function) normalizable vertex operators corresponding to both types of states described above to compute their S-matrix.

A very interesting feature of linear dilaton backgrounds is the existence of additional observables, corresponding to off-shell operators in the  $d$ -dimensional theory of the defect. These observables correspond to non-normalizable vertex operators which go like  $\exp(\beta\phi)$  with  $\beta > -Q/2$  as  $\phi \rightarrow \infty$ . Like in anti-de-Sitter space, correlation functions of these operators correspond to off-shell Green functions in LST. The main purpose of this paper is to elucidate the analytic structure of these correlation functions.

We will show that the off-shell Green functions of weakly coupled asymptotically linear dilaton string theory satisfy an analog of LSZ reduction, which is familiar from local quantum field theory<sup>1</sup>. They exhibit poles (which we will refer to as LSZ poles) at the

---

<sup>1</sup> Since LST is not a local Quantum Field Theory (QFT), the usual arguments for the LSZ reduction (see *e.g.* [11]) do not apply in this case.

locations of the normalizable states. The residues of these poles are on-shell correlation functions which involve the normalizable vertex operators creating these states from the vacuum; they can be used to study the interactions of these states.

Unlike local QFT, where all poles can be interpreted as due to particles going on-shell, it is well known that in asymptotically linear dilaton spacetimes there is another type of poles, associated with the contribution of the (semi-)infinite region  $\phi \rightarrow \infty$  in (1.3). Scattering processes that can occur uniformly at any value of  $\phi$  are enhanced by the volume of the  $\phi$  coordinate; this leads to poles in correlation functions. As we will see below, these poles play an important role in understanding the analytic structure of the LST correlation functions.

The off-shell Green functions of LST also exhibit other singularities, such as branch cuts associated with creating the continuum of scattering states mentioned above, and poles corresponding to intermediate states going on-shell. These singularities are rather standard, and we will not study them in detail in this paper.

Most of our discussion below will focus on a particular class of weakly coupled asymptotically linear dilaton spacetimes, which have the form (up to discrete identifications)

$$\mathbb{R}^{d-1,1} \times \frac{SL(2)_k}{U(1)} \times \tilde{\mathcal{M}}. \quad (1.4)$$

As before (1.3),  $\mathbb{R}^{d-1,1}$  labels the worldvolume of the defect;  $\frac{SL(2)_k}{U(1)}$  is the well studied ( $\mathcal{N} = 2$  supersymmetric) cigar CFT;  $\tilde{\mathcal{M}}$  is a compact CFT, closely related to  $\mathcal{M}$  in (1.3). The linear dilaton direction labeled by  $\phi$  (1.3) is in this case the direction along the cigar, and the boundary at  $\phi = \infty$  corresponds to the asymptotic region far from the tip of the cigar. The linear dilaton slope  $Q$  in (1.1) is related to the level of  $SL(2)$ ,  $k$ , via the relation

$$Q^2 = \frac{2}{k}. \quad (1.5)$$

In the  $SL(2)/U(1)$  theory the asymptotic form of the dilaton is given by (1.1), but the dilaton does not grow indefinitely; the string coupling reaches some maximal value  $g_s^{(tip)}$  at the tip of the cigar.

One motivation for studying the case (1.4) is that it arises naturally in the physics of  $NS5$ -branes and singularities of Calabi-Yau manifolds, where the process of going from (1.3) to (1.4) corresponds to smoothing the singularity or separating the fivebranes. In particular, the case  $d = 6$ ,  $\tilde{\mathcal{M}} = \frac{SU(2)_k}{U(1)}$  that will be of interest to us below, is obtained

by studying a system of parallel  $NS5$ -branes in type II string theory, spread out at equal distances on a circle in the transverse  $\mathbb{R}^4$ .

The background (1.4) describes a particularly symmetric deformation of the singularity, and thus is more tractable than the generic case. While some of the techniques we use are specific to this background, we expect most of the results we obtain to be much more general.

The dynamics of LST in spacetimes of the form (1.4) was discussed by [12,13,14,3] and others. In particular, in [13,14] a study of two and three-point functions was undertaken in the six dimensional LST corresponding to type IIB fivebranes distributed on a circle. The expectation was that at low energies the non-normalizable vertex operators in the background (1.4) should reduce to local, gauge-invariant operators in the low energy field theory on the branes, which in this case is an  $SU(k)$  gauge theory with sixteen supercharges ( $\mathcal{N} = (1, 1)$  supersymmetry in six dimensions), at a particular point in its Coulomb branch. The string theory correlation functions should reduce to off-shell Green functions of these operators.

Surprisingly, it was found that this is not the case. While the string theory correlation functions do exhibit poles which agree with gauge theory expectations, they also exhibit some additional poles. In particular, the string theory two-point functions have some unexpected poles at  $p_\mu^2 = 0$ . On general grounds, one expects such poles to signal the creation of massless single particle states from the vacuum. However, in the case of the additional poles found in [13,14] there were no candidate states in the  $U(1)^{k-1}$  gauge theory with the right quantum numbers. Also, the residues of some of the poles were negative, which in a particle interpretation would signal non-unitarity. This puzzling behavior motivated the work described in this paper.

Our general analysis of the correlation functions of non-normalizable vertex operators in backgrounds of the form (1.4) leads to the following explanation of the analytic structure of the amplitudes studied in [13,14]. Some of the poles of the two and three-point functions are of the LSZ type, and correspond to processes where the non-normalizable vertex operators act on the vacuum and create normalizable states on the cigar, that belong to the principal discrete series of  $SL(2)$ . The massless states of this type are in one to one correspondence with single particle states in the low energy field theory. As in the LSZ reduction in quantum field theory, the residue of the LSZ poles in a correlation function of off-shell observables may be interpreted as an S-matrix of single-particle states, or (equivalently) as a correlation function of normalizable observables.

All the poles found in [13,14] that are not expected from the low energy gauge theory analysis are of the bulk type. This seems surprising since such poles are associated with the infinite region far from the tip of the cigar, and one might expect this region to contribute only above the energy scale at which one can create states belonging to the continuum of scattering states that live there. Since this continuum starts (in the six dimensional example) above a gap of order  $M_s/\sqrt{k}$  (for  $k$  fivebranes), one might expect that it should not give rise to interesting effects at low energies. Nevertheless, as we will see, bulk effects lead to poles at  $p_\mu^2 = 0$ . This is a sort of UV/IR mixing (reminiscent of the mixing observed in non-commutative field theory [15,16]), where a massless pole is due to the contribution of massive states.

It is important to emphasize that if we restrict attention to the S-matrix of the massless single particle states, *i.e.* to correlation functions of the normalizable observables corresponding to the relevant principal discrete series states, we find a much more conventional picture. In particular, as far as is known, one can match the S-matrix elements computed from string theory in the background (1.4) with a standard low energy effective Lagrangian written in terms of the light fields. The bulk singularities do not arise in these correlation functions, which behave much like scattering amplitudes in other string theory backgrounds. The new element in asymptotically linear dilaton backgrounds is that off-shell observables make sense (unlike, say, in flat spacetime with a constant dilaton), and it is these observables that exhibit the new behavior.

Let us elaborate this point. LST provides us with a list of off-shell observables and their correlation functions. Taking all the momenta uniformly to zero in these correlation functions should normally lead one to a *conformal field theory*. On general grounds, we know that in any unitary six dimensional CFT, a scalar field  $\mathcal{O}(x_\mu)$  with scaling dimension two, which satisfies

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle \simeq \frac{c}{x^4} \quad (1.6)$$

is free and decoupled. Associated with it via the state-operator correspondence is a massless particle.

Thus, it is natural to argue that two-point functions which exhibit bulk poles at  $p_\mu^2 = 0$  should be interpreted as signalling the presence of extra massless particles in the theory. As mentioned above, there are at least two problems with this conclusion. One is that there are in fact no normalizable massless states with the right quantum numbers, either in the low energy gauge theory, or in the cigar description. The second is that some of the relevant two-point functions (1.6) have  $c < 0$ , in seeming contradiction with unitarity.

One possible response to this conundrum is that the limit taken to decouple the fivebranes from the bulk is inconsistent, and one cannot consistently consider the model in which the linear dilaton behavior (1.1) persists up to arbitrarily large  $\phi$ . Indeed, if one introduces a cutoff in the form of an upper bound on  $\phi$ , the offending bulk poles are smoothed out and replaced by large but finite contributions to the low energy Green functions.

We do not expect this resolution to be correct, for several reasons. First, models of the asymptotic form (1.3) seem to be well defined without such a cutoff. Second, we provide below a consistent interpretation of the analytic structure without appealing to the existence of a region outside the linear dilaton one. Third, general considerations of open-closed string duality to which we will return in §11.4 suggest that one might be able to reproduce the same analytic structure by studying D-branes localized in the strong coupling region (*e.g.* near the tip of the cigar (1.4)).

Instead, what seems to be going on here is a “violation of folklore” (to quote [17]). The UV and IR do not decouple, and one cannot describe the extreme IR behavior of LST correlation functions in terms of a standard six dimensional CFT. Clearly, it would be interesting to understand this phenomenon better.

The plan of this paper is as follows. We begin in section 2 with a detailed analysis of the analytic structure of correlation functions in spacetimes of the form (1.4), in the bosonic string. This is a useful warm up exercise for the superstring, which exhibits all the non-trivial elements that are important for our purposes. We show that in order to study the analytic structure of correlation functions on the cigar as a function of the external momenta, it is useful to write the operators in the coset CFT in terms of the natural observables in the underlying CFT on  $AdS_3$ ,  $\Phi_j(x, \bar{x})$ . Here,  $(x, \bar{x})$  label positions on the boundary of  $AdS_3$ , and in order to study observables on  $SL(2, \mathbb{R})/U(1)$  one has to perform a transform from  $(x, \bar{x})$  to the conjugate variables  $(m, \bar{m})$ . Correlation functions in the coset theory are naturally given by integrals over  $x_i$  of the corresponding correlation functions in CFT on  $AdS_3$ . This integral representation is very useful for studying the analytic structure of the amplitudes. In particular, we show that LSZ poles corresponding to principal discrete series states come from  $x_i \rightarrow 0, \infty$ , while bulk poles are associated with regions in the integrals over  $x_i$  where some or all of them approach each other.

In section 3 we briefly comment on the extension of the general results of section 2 to the superstring. Most of the analysis goes through, with only minor changes. In section 4 we discuss the six dimensional LST corresponding to  $k$   $NS5$ -branes spread out on a circle

in type IIB string theory. We find that the spectrum of massless principal discrete series states on the cigar is in one to one correspondence with the spectrum of single particle states in the low energy gauge theory.

In sections 5 and 6, we apply our results to the analysis of several correlation functions in the  $\mathcal{N} = (1, 1)$  supersymmetric  $d = 6$  LST mentioned above, and present the details of the picture described earlier in this introduction. In particular, we show that all the massless poles found in LST that do not correspond to states in the low energy gauge theory are bulk poles. Our improved understanding of the relation between the string theory and the low-energy gauge theory also allows us to find the precise mapping between string theory operators and low-energy gauge theory operators in this example, including the precise combination of single-trace and multi-trace operators that corresponds to single string vertex operators in the relevant geometry (1.4). As we vary the angular momentum of the operators in the compact space, we find that they change from single-trace operators, for very small angular momentum, to sub-determinant type operators when the angular momentum is close to its maximal value. This is in agreement with previous discussions of “giant gravitons” in the context of the AdS/CFT correspondence.

In sections 7-9 we comment on some additional applications of our results. Section 7 contains a brief discussion of four dimensional LSTs corresponding to isolated singularities of Calabi-Yau manifolds or equivalently wrapped fivebranes. In section 8 we discuss two dimensional bosonic string theory from the point of view of our analysis. In section 9 we discuss the thermodynamics of LSTs, and we argue that the Hagedorn temperature is a maximal temperature for LSTs. Section 10 contains a further discussion of our results, and section 11 mentions some future directions. Two appendices contain some useful technical results.

## 2. Bosonic string theory on $SL(2, \mathbb{R})/U(1)$

In this section we study the analytic structure of correlation functions in string theory on spacetimes of the form (1.4). As in many other instances in string theory, it is convenient to first consider the technically simpler bosonic case, and then generalize the discussion to the superstring (which we will do in the next sections).



## 2.1. Generalities

To make the transition to the superstring smoother we will denote the level of  $SL(2, \mathbb{R})$ , which the model possesses before modding out by  $U(1)$ , by  $k + 2$ . The central charge of the  $SL(2, \mathbb{R})/U(1)$  coset CFT is then given by

$$c_{sl} = \frac{3(k+2)}{k} - 1 = 2 + \frac{6}{k}. \quad (2.1)$$

With this definition, the slope of the linear dilaton  $Q$  in (1.1) is given by (1.5). The condition that (1.4) is a consistent background for the bosonic string takes the form

$$d + c_{sl} + c_{\tilde{\mathcal{M}}} = 26. \quad (2.2)$$

In studying perturbative string theory on the space (1.4), one is interested in computing correlation functions of physical vertex operators on this space. A large class of such operators, which will be sufficient for our purposes, can be constructed as follows.

Let  $\mathcal{W}$  be a conformal primary on  $\tilde{\mathcal{M}}$ , with scaling dimension  $(\Delta_L, \Delta_R)$ . We can form a physical vertex operator by “dressing”  $\mathcal{W}$  by an  $SL(2, \mathbb{R})/U(1) \times \mathbb{R}^{d-1,1}$  vertex operator,<sup>2</sup>  $V_{j;m,\bar{m}} e^{ip \cdot x}$ , to construct a physical observable<sup>3</sup>

$$\mathcal{O}_{\mathcal{W}}(p) = \mathcal{W} V_{j;m,\bar{m}} e^{ip \cdot x}. \quad (2.3)$$

The  $SL(2, \mathbb{R})/U(1)$  part of  $\mathcal{O}_{\mathcal{W}}$ ,  $V_{j;m,\bar{m}}$ , corresponds to a fundamental string carrying momentum and winding around the cigar. Its worldsheet scaling dimension is given by

$$\begin{aligned} \Delta_{j;m} &= -\frac{j(j+1)}{k} + \frac{m^2}{k+2}, \\ \bar{\Delta}_{j;\bar{m}} &= -\frac{j(j+1)}{k} + \frac{\bar{m}^2}{k+2}. \end{aligned} \quad (2.4)$$

At large  $\phi$ , the Euclidean cigar  $SL(2, \mathbb{R})/U(1)$  looks like a semi-infinite cylinder, and the operators  $V_{j;m,\bar{m}}$  have well-defined momentum  $n$  and winding number  $w$  ( $n, w \in \mathbf{Z}$ ) around the cylinder, given by

$$\begin{aligned} m &= \frac{1}{2} [w(k+2) + n], \\ \bar{m} &= \frac{1}{2} [w(k+2) - n]. \end{aligned} \quad (2.5)$$

---

<sup>2</sup> See appendix A for a description of vertex operators in  $SL(2, \mathbb{R})$  and  $SL(2, \mathbb{R})/U(1)$ .

<sup>3</sup> This is not the most general fundamental string excitation on (1.4) – we have suppressed the towers of transverse oscillators associated with  $\mathbb{R}^{d-1,1} \times SL(2, \mathbb{R})/U(1)$ .

The physical state condition reads

$$\frac{1}{2}p_\mu^2 + \Delta_{j;m} + \Delta_L = \frac{1}{2}p_\mu^2 + \bar{\Delta}_{j;\bar{m}} + \Delta_R = 1, \quad (2.6)$$

where  $\alpha' = 2$  and the signature convention is  $p_\mu^2 = p_i^2 - p_0^2$ . For given  $\mathcal{W}$ , and fixed momentum and winding  $(n, w)$  (or fixed  $(m, \bar{m})$  (2.5)), the physical state condition can be used to determine  $j$  as a function of  $p_\mu^2$ . For  $p_\mu^2$  above a certain critical value (which depends on  $\mathcal{W}, m, \bar{m}$ ),  $j$  is real and the operator  $V_{j;m,\bar{m}}$  is non-normalizable. On the other hand, for  $p_\mu^2$  smaller than that value, the solution of (2.6) is  $j = -\frac{1}{2} + i\lambda$ , with real  $\lambda$ . The resulting wave function is delta-function normalizable.

The dimension formula (2.4) is invariant under  $j \rightarrow -j - 1$ . In fact, the operators  $V_{j;m,\bar{m}}$  and  $V_{-j-1;m,\bar{m}}$  are related via the reflection property [18,19]

$$V_{j;m,\bar{m}} = R(j, m, \bar{m}; k) V_{-j-1;m,\bar{m}},$$

$$R(j, m, \bar{m}; k) = \nu(k)^{2j+1} \frac{\Gamma(1 - \frac{2j+1}{k}) \Gamma(j+m+1) \Gamma(j-\bar{m}+1) \Gamma(-2j-1)}{\Gamma(1 + \frac{2j+1}{k}) \Gamma(m-j) \Gamma(-j-\bar{m}) \Gamma(2j+1)}, \quad (2.7)$$

where<sup>4</sup>

$$\nu(k) = \frac{1}{\pi} \frac{\Gamma(1 + \frac{1}{k})}{\Gamma(1 - \frac{1}{k})}. \quad (2.8)$$

For real  $j$ , one can use (2.7) to restrict to  $j > -1/2$ , and we will usually do so below.

The relation (2.7) degenerates when the reflection coefficient  $R(j, m, \bar{m}; k)$  develops singularities. Poles of  $R$  have two sources that will be discussed below. One (associated with the first and last  $\Gamma$  functions in the numerator of the expression for  $R$ ) has to do with operator mixing due to bulk interactions of  $V_{j;m,\bar{m}}$  with the background. The other (associated with the factor  $\Gamma(j+m+1)\Gamma(j-\bar{m}+1)$  in (2.7)) signals the presence of normalizable states in the theory. Zeroes of  $R$ , associated with the second and third  $\Gamma$  functions in the denominator of (2.7), are related to the existence of degenerate operators in the theory (see *e.g.* §4 in [20]).

Correlation functions of the observables (2.3) are given by  $n$ -point functions

$$\langle \mathcal{O}_{\mathcal{W}_1}(p_1) \mathcal{O}_{\mathcal{W}_2}(p_2) \cdots \mathcal{O}_{\mathcal{W}_n}(p_n) \rangle, \quad (2.9)$$

---

<sup>4</sup> As explained in [20],  $\nu(k)$  is actually a free parameter, which depends on the couplings  $\lambda, \mu$  introduced in §2.4. The value of  $\nu(k)$  given here corresponds to the choice made in [18,19].

where  $(n - 3)$  vertex operators are integrated over the worldsheet<sup>5</sup>, and three are placed, as usual, at arbitrary points, say  $0, 1, \infty$ . When all the vertex operators in (2.9) obey  $j > -1/2$ , these amplitudes correspond in spacetime to off-shell Green functions in a  $d$ -dimensional LST. Our main focus here is on their analytic structure.

There are two sources of dependence on the momenta  $(p_1, \dots, p_n)$  in the correlation function (2.9). One is the standard dependence on the Mandelstam invariants  $p_i \cdot p_j$ , which comes from the correlation function of the  $\exp(ip_i \cdot x)$  factors in  $\mathcal{O}_{\mathcal{W}_i}$  (2.3). This contribution leads to singularities of the integrated correlation function, which occur when intermediate states go on mass-shell, and are well understood.

The second source of momentum dependence in (2.9) comes from the unintegrated  $SL(2, \mathbb{R})/U(1)$  correlation function,

$$\langle V_{j_1; m_1, \bar{m}_1} \cdots V_{j_n; m_n, \bar{m}_n} \rangle. \quad (2.10)$$

This  $n$ -point function has singularities as a function of the  $j_i$ , which correspond via (2.6) to singularities in momentum space. Our main purpose in this paper is to better understand these singularities, and in particular their spacetime interpretation.

Of special interest to us will be external leg poles, which occur as a single  $j$  (or  $p_\mu^2$  (2.6)) is tuned to a particular value. Below, we will exhibit the origin of these poles, and show that they are always *single* poles, which are associated with asymptotic single-particle states.

These poles can be used to implement an analog of the LSZ reduction in LST; thus we will refer to them below as LSZ poles. As one approaches an LSZ pole, we will see that one has

$$\mathcal{O}_{\mathcal{W}}(p) \sim \frac{1}{p^2 + M^2} \mathcal{O}_{\mathcal{W}}^{(\text{norm})}(p), \quad (2.11)$$

where  $\mathcal{O}_{\mathcal{W}}^{(\text{norm})}$  is a normalizable vertex operator, which creates from the vacuum a particle with mass  $M$ . In the bosonic string,  $M^2$  can be negative. In spacetime supersymmetric theories it will always be non-negative.

Near a pole (2.11) as a function of one of the momenta in (2.9), say  $p_n$ , the  $n$ -point function has the form

$$\langle 0 | \mathcal{O}_{\mathcal{W}_1}(p_1) \mathcal{O}_{\mathcal{W}_2}(p_2) \cdots \mathcal{O}_{\mathcal{W}_n}(p_n) | 0 \rangle \simeq \frac{1}{p_n^2 + M_n^2} \langle 0 | \mathcal{O}_{\mathcal{W}_1}(p_1) \cdots \mathcal{O}_{\mathcal{W}_{n-1}}(p_{n-1}) | \mathcal{W}_n, p_n \rangle \quad (2.12)$$

---

<sup>5</sup> Although many of our considerations below are more general, we will mainly discuss in this paper tree-level string theory, *i.e.* a worldsheet with spherical topology.

where  $|\mathcal{W}_n, p_n\rangle = \mathcal{O}_{\mathcal{W}_n}^{(\text{norm})}(p_n)|0\rangle$  is the state created by the normalizable operator (2.11) acting on the vacuum.

As we approach poles of the form (2.11) in all the external momenta  $p_i$  in (2.9), the amplitude behaves as

$$\langle \mathcal{O}_{\mathcal{W}_1}(p_1) \cdots \mathcal{O}_{\mathcal{W}_n}(p_n) \rangle \sim \left( \prod_i \frac{1}{p_i^2 + M_i^2} \right) \langle 0 | \mathcal{O}_{\mathcal{W}_1}^{(\text{norm})}(p_1) \cdots \mathcal{O}_{\mathcal{W}_n}^{(\text{norm})}(p_n) | 0 \rangle. \quad (2.13)$$

As in QFT, the residue of the poles – the  $n$ -point function of the normalizable operators  $\mathcal{O}_{\mathcal{W}_i}^{(\text{norm})}(p_i)$  – is proportional to the S-matrix of the particles with masses  $M_i$  created by these operators from the vacuum<sup>6</sup>.

In the next two subsections we will exhibit the origin of the poles (2.11) in LST backgrounds of the form (1.4). Later, we will also discuss other singularities of the amplitudes (2.10) that have a different origin and play an important role in some applications.

## 2.2. LSZ in bosonic LST: (I) a semiclassical analysis

In order to exhibit the poles (2.11) we have to understand better the structure of the  $SL(2, \mathbb{R})/U(1)$  operators  $V_{j;m,\bar{m}}$  which give rise to them. One way to construct these operators is to start with the corresponding operators  $\Phi_{j;m,\bar{m}}$  in CFT on the  $SL(2, \mathbb{R})$  group manifold, and remove from them the  $U(1)$  part. In this subsection we use a semiclassical analysis to do this. In the next subsection we generalize it to the full quantum worldsheet theory.

The  $SL(2, \mathbb{R})$  group manifold can be thought of as Minkowski  $AdS_3$ . For our purposes, it is convenient to analytically continue to Euclidean  $AdS_3$ , which we can parametrize by the Poincaré coordinates  $(\phi, \gamma, \bar{\gamma})$ , with the metric

$$ds^2 = d\phi^2 + e^{Q\phi} d\gamma d\bar{\gamma}. \quad (2.14)$$

A natural set of observables in CFT on Euclidean  $AdS_3$  is given by the eigenfunctions of the Laplacian [18,19]

$$\Phi_j(x, \bar{x}) = \frac{1}{\pi} \left( |\gamma - x|^2 e^{\frac{Q\phi}{2}} + e^{-\frac{Q\phi}{2}} \right)^{-2(j+1)}, \quad (2.15)$$

---

<sup>6</sup> To obtain the S-matrix precisely one needs to suitably normalize the states  $|\mathcal{W}_i, p_i\rangle$ . Note that apriori one might have thought that the poles (2.13) themselves were just an artifact of choosing a wrong normalization for the vertex operators  $\mathcal{O}_{\mathcal{W}}(p)$ , but our discussion will make it clear that this is not the case.

with  $j > -\frac{1}{2}$ . The parameters  $(x, \bar{x})$ , on which these observables depend, label positions on the boundary of  $AdS_3$ . The operators (2.15) transform as primaries of dimension  $h = \bar{h} = j + 1$  under conformal transformations of this boundary. The coordinates  $(x, \bar{x})$  should not be confused with the string worldsheet coordinates  $(z, \bar{z})$ , which are mostly suppressed in this paper.

For the application to the coset theory it will be useful to expand  $\Phi_j$  around  $\phi = \infty$  [21],

$$\Phi_j(x, \bar{x}) \simeq \frac{1}{2j+1} e^{Qj\phi} \delta^2(\gamma - x) + O(e^{Q(j-1)\phi}) + \frac{e^{-Q(j+1)\phi}}{\pi|\gamma - x|^{4(j+1)}} + O(e^{-Q(j+2)\phi}). \quad (2.16)$$

For generic  $j$ , the large  $\phi$  expansion naturally splits into two independent series. One includes the leading term,  $e^{Qj\phi} \delta^2(\gamma - x)$  and an infinite series of corrections of the form  $e^{Q(j-n)\phi} \partial_x^n \bar{\partial}_{\bar{x}}^n \delta^2(\gamma - x)$ . This series is relevant only near  $\gamma = x$ . The second series starts with the dominant term for generic  $\gamma$ ,  $e^{-Q(j+1)\phi} |\gamma - x|^{-4(j+1)}$  and includes corrections that are down by powers of  $e^{-Q\phi} / |\gamma - x|^2$ . This series consists purely of terms which decay exponentially as  $\phi \rightarrow \infty$ , and are thus normalizable there.

The  $AdS_3$  ancestor of the  $SL(2, \mathbb{R})/U(1)$  observable  $V_{j;m,\bar{m}}$  discussed above is the operator

$$\Phi_{j;m,\bar{m}} \equiv \int d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_j(x, \bar{x}). \quad (2.17)$$

Plugging the large  $\phi$  expansion (2.16) into (2.17) we find:

$$\begin{aligned} \Phi_{j;m,\bar{m}} &= \frac{1}{2j+1} e^{Qj\phi} \gamma^{j+m} \bar{\gamma}^{j+\bar{m}} + O(e^{Q(j-1)\phi}) + \\ &\quad \frac{\Gamma(j+m+1)\Gamma(j-\bar{m}+1)\Gamma(-2j-1)}{\Gamma(m-j)\Gamma(-j-\bar{m})\Gamma(2j+2)} e^{-Q(j+1)\phi} \gamma^{m-j-1} \bar{\gamma}^{\bar{m}-j-1} + O(e^{-Q(j+2)\phi}). \end{aligned} \quad (2.18)$$

In computing the second line of (2.18) we have rescaled  $x$  by  $\gamma$  and used the result

$$\int d^2x x^{j+m} \bar{x}^{j+\bar{m}} |1-x|^{-4(j+1)} = \pi \frac{\Gamma(j+m+1)\Gamma(j-\bar{m}+1)\Gamma(-2j-1)}{\Gamma(m-j)\Gamma(-j-\bar{m})\Gamma(2j+2)}. \quad (2.19)$$

Using standard techniques (see *e.g.* [22]) one can write the vertex operator (2.18) as a product of a plane wave living in the  $U(1)$  CFT and an operator in the coset, whose asymptotic form far from the tip of the cigar is

$$V_{j;m,\bar{m}} = \frac{e^{i\sqrt{\frac{2}{k+2}}(mY - \bar{m}\bar{Y})}}{2j+1} \left[ e^{Qj\phi} + \frac{\Gamma(j+m+1)\Gamma(j-\bar{m}+1)\Gamma(-2j-1)}{\Gamma(m-j)\Gamma(-j-\bar{m})\Gamma(2j+1)} e^{-Q(j+1)\phi} + \dots \right]. \quad (2.20)$$

Here  $Y$  is the angular coordinate around the cigar. It lives on a circle of radius  $\sqrt{2(k+2)}$ .

We see that (2.20) has the structure anticipated in (2.11). The leading non-normalizable<sup>7</sup> contribution to the vertex operator is finite, while the leading normalizable term has poles at various values of  $j$ . Note that the relative coefficient of the  $e^{-Q(j+1)\phi}$  and  $e^{Qj\phi}$  terms in (2.20) coincides with the large  $k$  limit of the reflection coefficient  $R$  in (2.7) (the power of  $\nu(k)$  in  $R$  is unimportant since we can generate it by shifting  $\phi \rightarrow \phi + \phi_0$  in (2.20)). At finite  $k$ , the ratio of  $\Gamma$  functions in (2.20) should indeed be replaced by the exact reflection coefficient (2.7).

An important point is that not all of the poles in (2.20) give rise to normalizable states as indicated in (2.11). To see which ones do and which ones do not, it is convenient to go back to the integral representation (2.19) that gave rise to the second term in (2.20). The poles associated with the three  $\Gamma$  functions in the numerator come from the vicinity of  $x = 0$ ,  $x = \infty$  and  $x = 1$ , respectively. Let us consider these singularities in turn, starting with the region  $x \rightarrow 0$ . The contribution of this region to the integral (2.17) is given by

$$\int_{|x|<\epsilon} d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_j(x, \bar{x}) \quad (2.21)$$

where  $\epsilon \ll 1$  is a cutoff, whose role is to isolate the singularity arising from  $x = 0$ . We will see shortly that the integral (2.21) has poles at

$$j = M - 1, M - 2, \dots > -\frac{1}{2}, \quad M = \min\{|m|, |\bar{m}|\}, \quad m, \bar{m} < -\frac{1}{2}. \quad (2.22)$$

Note that this is the same as the set of poles of the second term in (2.20) associated with the first  $\Gamma$  function in the numerator, as advocated above. We will see that the residues of the poles agree with (2.18), (2.20) as well.

To prove (2.22), we assume (without loss of generality) that the momentum on the cigar

$$n \equiv m - \bar{m} \quad (2.23)$$

is a non-negative integer. If  $n < 0$ , one can interchange the roles of  $(x, m) \leftrightarrow (\bar{x}, \bar{m})$  and repeat the analysis below.

We may rewrite (2.21) as

$$\int_{|x|<\epsilon} d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_j(x, \bar{x}) = \int_{|x|<\epsilon} d^2x |x|^{2(j+m)} \bar{x}^{-n} \Phi_j(x, \bar{x}). \quad (2.24)$$

---

<sup>7</sup> Recall that  $j > -1/2$ .

The leading singularity of (2.24) is obtained as follows. Using the fact that  $\Phi_j(x, \bar{x})$  (2.15) is analytic at  $x = 0$  and expanding it in powers of  $x, \bar{x}$ , we see that the first term that survives the integral over the phase of  $x$  is  $\Phi_j(x, \bar{x}) = \cdots + \frac{1}{n!} \bar{\partial}^n \Phi_j(0) \bar{x}^n + \cdots$ . Its contribution to (2.24) is

$$\frac{1}{n!} \int_{|x| < \epsilon} d^2 x |x|^{2(j+m)} \bar{\partial}^n \Phi_j(0) . \quad (2.25)$$

This integral diverges when  $j + m \rightarrow -1$ , where it has a simple pole:

$$\int_{|x| < \epsilon} d^2 x |x|^{2(j+m)} \bar{\partial}^n \Phi_j(0) \simeq \frac{\pi \bar{\partial}^n \Phi_j(0)}{j + m + 1} . \quad (2.26)$$

For  $j + m < -1$  the integral (2.24) is also divergent and should be treated via analytic continuation, as is familiar from studies of Shapiro-Virasoro amplitudes in string theory. To study the subleading singularities, one needs to expand  $\Phi_j(x, \bar{x})$  to  $l$ 'th order in  $x$ , and  $(n + l)$ 'th order in  $\bar{x}$ , and consider the resulting integral:

$$\int_{|x| < \epsilon} d^2 x |x|^{2(j+m+l)} \partial^l \bar{\partial}^{n+l} \Phi_j(0) \simeq \frac{\pi \partial^l \bar{\partial}^{n+l} \Phi_j(0)}{j + m + l + 1} . \quad (2.27)$$

We conclude that for  $m \geq \bar{m}$ , the singularities of (2.24) occur at

$$j + m + l + 1 = 0; \quad l = 0, 1, 2, \dots . \quad (2.28)$$

Since  $j > -\frac{1}{2}$ , we see that (2.28) has solutions iff  $m < -\frac{1}{2}$ , and since  $m \geq \bar{m}$  this also implies  $\bar{m} < -\frac{1}{2}$ . This concludes the proof of (2.22).

We see that as  $j$  approaches one of the values (2.22), the vertex operator  $V_{j;m,\bar{m}}$  develops a pole associated with the  $\Gamma(j + m + 1)$  factor in (2.20). The residue of this pole is a normalizable operator, as indicated in (2.11). This operator creates from the vacuum a physical state in the spacetime theory. It can be obtained by starting with the operator  $\partial^l \bar{\partial}^{n+l} \Phi_j(0)$  in CFT on  $AdS_3$ , and removing the  $U(1)$  part.

The region  $x \rightarrow \infty$  can be studied in a very similar way. In this limit, the vertex operator (2.15) behaves as

$$\Phi_j(x, \bar{x}) \propto |x|^{-4(j+1)} . \quad (2.29)$$

Thus, the leading singularities from this region are due to the behavior of the integral

$$\int_{|x| > 1/\epsilon} d^2 x x^{j+m} \bar{x}^{j+\bar{m}} |x|^{-4(j+1)} , \quad (2.30)$$

while subleading singularities receive contributions from subleading terms in the large  $x$  expansion of  $\Phi_j$ . A similar analysis to the previous case leads to poles at

$$j = M - 1, M - 2, \dots > -\frac{1}{2}, \quad M = \min\{m, \bar{m}\}, \quad m, \bar{m} > \frac{1}{2}. \quad (2.31)$$

This is the same set of poles as that associated with the  $\Gamma(j - \bar{m} + 1)$  factor on the right-hand side of (2.20), in agreement with our comments above. One can check that the residues agree as well.

Note that the physical states (2.22), (2.31) appear only when the signs of  $m$  and  $\bar{m}$  are the same. Looking back at equation (2.5) we see that this means that these states are “winding dominated.” As we will discuss later (in §10.1), this is very natural from the spacetime perspective. Winding modes feel an attractive potential towards the tip, while momentum modes feel a repulsive one [23]. Thus, fundamental string modes that are dominated by winding can bind to the tip of the cigar. The normalizable states corresponding to the poles (2.22), (2.31) are precisely such bound states.

So far we have discussed the first two  $\Gamma$  functions in the numerator of the second term in (2.20), attributed them to the contributions from  $x = 0, \infty$  in the integral (2.19), and interpreted them as corresponding to normalizable physical states created from the vacuum by the operators  $V_{j;m,\bar{m}}$ , as in the discussion following equation (2.11).

The third  $\Gamma$  function in (2.20),  $\Gamma(-2j - 1)$ , leads to divergences at  $2j + 1 = 0, 1, 2, \dots$ . These can be seen to arise from the region  $x \rightarrow 1$  in the integral (2.19), or before rescaling  $x$  by  $\gamma$ , from  $x \rightarrow \gamma$ . Interestingly, these poles do not signal the appearance of additional physical states. Instead, they signal the breakdown of the large  $\phi$  expansion (2.16). Indeed, the expansion parameter for  $x \neq \gamma$  in (2.16),  $e^{-Q\phi}/|\gamma - x|^2$ , becomes large when  $x \simeq \gamma$ . In order to evaluate the contribution of this region to the integral (2.17) we have to sum the full expansion indicated in (2.16). A related complication is that the two series in the large  $\phi$  expansion (2.16), both of which are important for  $x \simeq \gamma$ , mix when  $j$  takes half-integer values. We will see later (in §2.4) that the behavior of the vertex operator (2.20) near the poles at  $2j + 1 \in \mathbf{Z}_+$  is in general dominated by non-trivial interactions that occur at large positive  $\phi$ .

At any rate, to see the actual behavior as  $x \rightarrow \gamma$ , we substitute (2.15) in (2.17) and evaluate

$$\frac{1}{\pi} \int_{|x-\gamma|<\epsilon} d^2x x^{j+m} \bar{x}^{j+\bar{m}} \left( |\gamma - x|^2 e^{\frac{Q\phi}{2}} + e^{-\frac{Q\phi}{2}} \right)^{-2(j+1)}. \quad (2.32)$$



This integral is actually finite for positive integer  $2j+1$ . We conclude that the contribution of the region  $x \simeq \gamma$  to the integral (2.17) is finite. The apparent singularities of  $\Gamma(-2j-1)$  in (2.20) do not in fact lead to a divergence of the full vertex operator, and do not correspond to physical states via (2.11).

To summarize, we see that at least semiclassically, the observables (2.3) indeed exhibit the behavior (2.11) near poles determined by the physical state conditions (2.4) – (2.6) and (2.22), (2.31). The normalizable vertex operators  $\mathcal{O}_{\mathcal{W}}^{(\text{norm})}$  behave as  $\phi \rightarrow \infty$  (far from the tip of the cigar) like  $\exp(-Q(j+1)\phi)$ , with  $j$  given by the appropriate value (2.22), (2.31). A fact that will be useful below is that in the representation of these operators in terms of observables in  $AdS_3$  integrated over the boundary variables  $(x, \bar{x})$ , these poles come from the regions  $x \rightarrow 0, \infty$ .

### 2.3. LSZ in bosonic LST: (II) exact results

In the previous subsection we studied the behavior of the vertex operators  $V_{j;m,\bar{m}}$  semiclassically. In particular, we treated the  $AdS_3$  operators  $\Phi_j$  (2.15) as functions and discussed their properties. This provides a good approximation in the limit  $k \rightarrow \infty$  ( $k$  is related to the level of  $SL(2, \mathbb{R})$ , see equation (2.1)), in which the  $\sigma$ -model on the cigar becomes weakly coupled and fluctuations are small. At finite  $k$ , the operators  $\Phi_j$  and  $V_{j;m,\bar{m}}$  should be thought of as fluctuating quantum operators, and cannot be treated as functions. Nevertheless, as we will see in this subsection, the results of the previous subsection can be extended to the quantum theory, with only minor modifications.

In the full quantum worldsheet theory, the  $AdS_3$  observables (2.15), (2.17) correspond to local operators with worldsheet scaling dimension

$$\Delta_j = \bar{\Delta}_j = -\frac{j(j+1)}{k} . \quad (2.33)$$

Removing from them the  $U(1)$  part, one finds the observables on the cigar,  $V_{j;m,\bar{m}}$ , whose scaling dimensions are given by (2.4).

The data that we have access to in the full CFT on the cigar is the set of worldsheet correlation functions (2.10). The symmetries of the problem imply that momentum on the cigar is conserved, *i.e.* (see (2.5))  $\sum_i n_i = \sum_i (m_i - \bar{m}_i) = 0$ . Winding, on the other hand, need not be conserved, since strings wound around the cigar can unwind at the tip. While winding violating amplitudes are not much more difficult to study, in this paper we will (for simplicity) focus on amplitudes that conserve winding. We will comment on the more general case in §2.6 below.

Imposing both momentum and winding conservation leads to the constraint

$$\sum_i m_i = \sum_i \bar{m}_i = 0. \quad (2.34)$$

In this case, we can compute the amplitude (2.10) directly in the  $SL(2, \mathbb{R})$  (or  $AdS_3$ ) CFT, since

$$\langle V_{j_1; m_1, \bar{m}_1} \cdots V_{j_n; m_n, \bar{m}_n} \rangle = C_{1 \dots n} \langle \Phi_{j_1; m_1, \bar{m}_1} \cdots \Phi_{j_n; m_n, \bar{m}_n} \rangle \quad (2.35)$$

where  $C_{1 \dots n}$  is a known function of the moduli (including the positions of the operators on the worldsheet), coming from a correlation function in a  $U(1)$  CFT. For the purpose of our discussion here, it gives an uninteresting overall constant.

Using (2.17) we see that the  $SL(2, \mathbb{R})/U(1)$  correlator (2.35) can be written as

$$\langle V_{j_1; m_1, \bar{m}_1} \cdots V_{j_n; m_n, \bar{m}_n} \rangle = C_{1 \dots n} \prod_{i=1}^n \int d^2 x_i x_i^{j_i + m_i} \bar{x}_i^{j_i + \bar{m}_i} \langle \Phi_{j_1}(x_1, \bar{x}_1) \cdots \Phi_{j_n}(x_n, \bar{x}_n) \rangle. \quad (2.36)$$

The semiclassical analysis of the previous subsection leads us to expect that the contributions of the regions  $x_i \rightarrow 0, \infty$  in (2.36) give rise to poles at values of  $j_i$  corresponding to (2.22), (2.31), respectively.

Consider, for example, the region  $x_1 \rightarrow 0$  in the  $x_1$  integral in (2.36). The correlator  $\langle \Phi_{j_1}(x_1, \bar{x}_1) \cdots \Phi_{j_n}(x_n, \bar{x}_n) \rangle$  in  $SL(2, \mathbb{R})$  is analytic<sup>8</sup> near  $x_1 = 0$  with fixed generic  $(x_2, \dots, x_n)$ . This can be seen by thinking about it as an expectation value in an AdS/CFT-dual conformal field theory in  $x$  space – singularities are expected to occur only when some of the  $x_i$  coincide, and there is nothing special about the point  $x_1 = 0$ . Therefore, the analysis following equation (2.21) is directly applicable, and leads to the same conclusions as there: the region near  $x_1 = 0$  in the integral (2.36) gives rise to singularities when  $(j_1, m_1)$  are related as in (2.22). For example, for  $m_1 = \bar{m}_1 < -1/2$ , the leading singularity is at  $j_1 = -m_1 - 1$ , near which (2.36) behaves as (compare to (2.12)):

$$\langle V_{j_1; m_1, \bar{m}_1} \cdots V_{j_n; m_n, \bar{m}_n} \rangle \simeq \frac{\pi C_{1 \dots n}}{j_1 + m_1 + 1} \times \prod_{i=2}^n \int d^2 x_i x_i^{j_i + m_i} \bar{x}_i^{j_i + \bar{m}_i} \langle \Phi_{j_1}(0) \Phi_{j_2}(x_2, \bar{x}_2) \cdots \Phi_{j_n}(x_n, \bar{x}_n) \rangle. \quad (2.37)$$

---

<sup>8</sup> In our discussion here (and below) we implicitly assume that the worldsheet locations  $z_i$  of the operators are generic.

Note that (as discussed above) the operator  $\Phi_{j_1}(0)$  in (2.37) is a normalizable vertex operator which thus corresponds to an asymptotic state,  $\langle 0|\Phi_{j_1}(0)$ . This state exists already in  $AdS_3$ , and focusing on the residue of the pole (2.37) picks out its contribution to the correlation function (2.36). Subleading singularities can be analyzed as in (2.27); they correspond to the asymptotic states  $\langle 0|\partial^l\bar{\partial}^{\bar{l}}\Phi_j(0)$ .

At the same time that  $x_1 \rightarrow 0$ , we can also send another of the  $x_i$ , say  $x_n$ , to infinity. The  $SL(2, \mathbb{R})$  Ward identities imply that as  $x_n \rightarrow \infty$ , the correlation functions of the  $\Phi_{j_i}$  in (2.36) behave as<sup>9</sup>

$$\langle \Phi_{j_1}(x_1, \bar{x}_1) \cdots \Phi_{j_n}(x_n, \bar{x}_n) \rangle \propto |x_n|^{-4(j_n+1)} \langle \Phi_{j_1}(x_1, \bar{x}_1) \cdots \Phi_{j_n}(\infty) \rangle. \quad (2.38)$$

This can again be compared to the semiclassical result (2.29), and proceeding as there one concludes that the region  $x_n \rightarrow \infty$  gives rise to potential poles at (2.31). For example, for  $m_n = \bar{m}_n > 1/2$ , the leading singularity is at  $j_n = m_n - 1$ .

The joint contribution of  $x_1 \rightarrow 0$  and  $x_n \rightarrow \infty$  behaves near the leading LSZ poles as

$$\begin{aligned} \langle V_{j_1; m_1, \bar{m}_1} \cdots V_{j_n; m_n, \bar{m}_n} \rangle &\simeq \frac{\pi^2 C_{1 \dots n}}{(m_1 + j_1 + 1)(m_n - j_n - 1)} \times \\ &\prod_{i=2}^{n-1} \int d^2 x_i x_i^{j_i + m_i} \bar{x}_i^{j_i + \bar{m}_i} \langle 0 | \Phi_{j_1}(0) \Phi_{j_2}(x_2, \bar{x}_2) \cdots \Phi_{j_{n-1}}(x_{n-1}, \bar{x}_{n-1}) \Phi_{j_n}(\infty) | 0 \rangle. \end{aligned} \quad (2.39)$$

The residue of the poles in (2.39) can be interpreted as the expectation value of  $n - 2$  off-shell operators between two normalizable states,  $\langle 0 | \Phi_{j_1}(0)$  and  $\Phi_{j_n}(\infty) | 0 \rangle$ .

Suppose we want to take more of the external legs on-shell, *e.g.* to compute the S-matrix, as in (2.13). Naively, there is a problem, since after sending one of the  $x_i$  in (2.36) to zero, and one to infinity, if we try to send additional  $x_i$  to zero or infinity, they will in particular approach  $x_1$  or  $x_n$ , which will spoil the above analysis. For example, it is no longer true that the correlator  $\langle \Phi_{j_1}(0) \Phi_{j_2}(x_2, \bar{x}_2) \cdots \Phi_{j_{n-1}}(x_{n-1}, \bar{x}_{n-1}) \Phi_{j_n}(\infty) \rangle$  is regular as  $x_2 \rightarrow 0$ , due to the short distance singularities of  $\Phi_{j_2}(x_2, \bar{x}_2)$  and  $\Phi_{j_1}(0)$ .

The problem is evidently due to interactions between the two asymptotic states  $\langle 0 | \Phi_{j_1}(0)$  and  $\langle 0 | \Phi_{j_2}(0)$ . In order to study this issue we need to understand the behavior of

$$\langle 0 | \Phi_{j_1}(0) \Phi_{j_2}(x_2, \bar{x}_2) \quad (2.40)$$

---

<sup>9</sup> On the right hand side of this formula, and below, we define  $\Phi_j(\infty) \equiv \lim_{x \rightarrow \infty} |x|^{4(j+1)} \Phi_j(x)$ ; it is easy to see that this limit is well-defined in any correlation function by using the conformal transformation  $x \rightarrow 1/x$ .

in the limit  $x_2 \rightarrow 0$ . This OPE<sup>10</sup> was analyzed in [19,24], where it was found that it receives two types of contributions. One is given by an integral over the delta function normalizable states corresponding to  $\Phi_j$  with  $j = -\frac{1}{2} + i\lambda$ . This contribution has to do with interactions between the states associated with  $\Phi_{j_1}(0)$  and  $\Phi_{j_2}(0)$ . It gives a different kinematic structure from what we are looking for here; we will return to contributions of this type in the next subsection.

In addition to the integral over continuous series states, the OPE (2.40) contains a discrete sum over states with real  $j$  [19,24]. This contribution is given by a power series in  $x_2, \bar{x}_2$ . The leading term goes like  $|x_2|^0$ ; its coefficient can be thought of as the two particle state  $\langle 0 | \Phi_{j_1} \Phi_{j_2}(0) \rangle$  (where the operator  $\Phi_{j_1} \Phi_{j_2}(0)$  is defined as the operator appearing in the  $|x_2|^0$  term in the OPE; it is a regularized product of the two operators). Higher order terms correspond to states of the form  $\langle 0 | \Phi_{j_1} \partial^n \bar{\partial}^{\bar{n}} \Phi_{j_2}(0) \rangle$ . In our analysis, these contributions give rise to poles of (2.36) coming from  $x_2 \rightarrow 0$ , with the leading pole, corresponding to the state  $\langle 0 | \Phi_{j_1} \Phi_{j_2}(0) \rangle$ , occurring at  $j_2 = -m_2 - 1$ , etc.

We expect a similar analysis to apply when we send any number of operators to  $x = 0$  and/or to  $x = \infty$ . Thus, we see that the correlation function (2.36) indeed has the structure expected from the LSZ reduction. The regions  $x_i \rightarrow 0, \infty$  give poles in the  $j_i$ , at the locations (2.22), (2.31) respectively<sup>11</sup>. The residues of these poles are matrix elements involving the normalizable vertex operators creating the relevant states from the vacuum.

We finish this subsection with some comments:

- (1) We see that a vertex operator of the form (2.3) can create many different normalizable states when acting on the vacuum as in (2.11), corresponding to different values of  $j$  in (2.22), (2.31). Thus, in general the mapping between states and operators is not one to one. This is standard in non-conformal theories, and for example is expected to be generically true in large  $N$  confining gauge theories. Generally, even if we normalize the operator such that it creates a particular one-particle state with canonical normalization, other states it creates will not be canonically normalized. These normalizations need to be taken into account when extracting the S-matrix from the correlation functions using (2.13).

---

<sup>10</sup> Note that here we are discussing the OPE on the boundary of  $AdS_3$  parametrized by the coordinates  $(x, \bar{x})$  and not on the string worldsheet, which is labeled by  $(z, \bar{z})$ .

<sup>11</sup> There is a slight subtlety associated with the last of the  $n$  poles in the  $n$ -point function, which we will briefly discuss in §2.5.

- (2) Like in QFT, the behavior (2.13) appears only in  $n$ -point functions with  $n \geq 3$ . The two point function has single poles, rather than the double poles implied by (2.13). We will exhibit this and discuss it further in some examples in §5.
- (3) An interesting feature of the discussion of this section is the role played by the parameters  $x_i$  (see *e.g.* (2.36)) in the analysis of the analytic structure of the amplitudes. The  $x_i$  can be thought of as complexified Schwinger parameters in the spacetime theory. In QFT, one way to introduce the Schwinger parameter (or proper time) is by replacing

$$\frac{1}{p^2 + M^2} \rightarrow \int_0^\infty dt e^{-t(p^2 + M^2)}. \quad (2.41)$$

The divergence as  $p^2 \rightarrow -M^2$  is due to the region of large Schwinger parameter. In our discussion in the previous section, the role of the Schwinger parameter is played by  $\pm \log |x|$  (see *e.g.* (2.26), (2.30)).

- (4) A new effect that needs to be taken into account at finite  $k$  is the upper bound on  $j$ ,  $j < (k-1)/2$  [13,24]. In string theory on  $AdS_3$ , operators  $\Phi_j$  with  $j > (k-1)/2$  are expected not to exist, and the coset should presumably inherit this bound. The role of this bound in string theory on the cigar is not well understood, and we will not discuss it further here.

#### 2.4. Bulk poles

So far we have restricted our attention to poles of the  $n$ -point function (2.10) that occur as we tune a particular  $j$  (or  $p_\mu^2$  (2.6)) to specific values. We identified the origin of such poles in the  $SL(2, \mathbb{R})/U(1)$  CFT and interpreted them as associated with the analog of LSZ reduction in string theory on the cigar.

The correlation functions (2.10) have another class of singularities that plays an important role in understanding string dynamics on spaces of the form (1.4), to which we turn next. These singularities are due to the infinite length of the cigar and are associated with processes that can occur arbitrarily far from the tip.

To explain the origin of these singularities, we next recall some features of CFT on asymptotically linear dilaton spaces, such as Liouville theory and  $SL(2, \mathbb{R})/U(1)$  (see *e.g.* [25] for a more detailed discussion), and contrast them with the more familiar case of CFT on flat space.

Let  $y$  be a non-compact scalar field on the worldsheet, *e.g.* one of the spatial directions in  $\mathbb{R}^{d-1}$ . The analog of the correlator (2.10) for it is the  $n$ -point function  $\langle e^{ip_1 y} \dots e^{ip_n y} \rangle$ .

Translation invariance implies that this correlator is proportional to  $\delta(p_1 + \dots + p_n)$ , *i.e.* it vanishes when the sum of momenta is non-zero, and is infinite when it is zero. The infinity is interpreted as the volume (or length) of  $y$ -space. Physically, it appears since the process computed by this correlator can occur at any  $y$ , with an amplitude that does not depend on  $y$ . We are interested in the amplitude per unit volume (for example, this is what enters the calculation of the S-matrix in flat spacetime); hence, this infinity is usually factored out.

Replacing  $y$  by an asymptotically linear dilaton direction  $\phi$  (1.1), such as the radial coordinate along the cigar, leads to a different picture. The dynamics is no longer translationally invariant, both because of the non-trivial dilaton and due to whatever effects resolve the strong coupling singularity at  $\phi = -\infty$ . In the case of the cigar, this is the metric, which depends non-trivially on  $\phi$ .

Thus, there is no longer any reason for the amplitudes to be proportional to the length of  $\phi$ -space, and indeed, in general they are not. In fact, since the string coupling goes to zero far from the tip of the cigar, correlation functions are typically dominated by the vicinity of the tip.

By tuning the “momenta along the cigar”  $j_i$  (2.10) one can reach resonances, or bulk amplitudes, which are processes that *can* occur anywhere along the cigar with uniform amplitude and are thus enhanced by the length of  $\phi$ -space. For example, keeping only the leading terms in the large  $\phi$  expansion of the  $V_{j;m,\bar{m}}$ ’s (2.20), and focusing on the  $\phi$  dependence, we have

$$\langle V_{j_1;m_1,\bar{m}_1} \cdots V_{j_n;m_n,\bar{m}_n} \rangle \sim \langle e^{Qj_1\phi} \cdots e^{Qj_n\phi} \rangle. \quad (2.42)$$

If the  $j_i$  satisfy the “anomalous momentum conservation condition”

$$\sum_i j_i = -1, \quad (2.43)$$

the amplitude can be shown to diverge like the volume of  $\phi$ -space, as in the flat space case mentioned above. One can think of this divergence as coming from the integral over the zero mode of the field  $\phi$ . However, unlike the flat space case, here the amplitude (2.42) is non-zero in the vicinity of the surface (2.43); it behaves like

$$\langle V_{j_1;m_1,\bar{m}_1} \cdots V_{j_n;m_n,\bar{m}_n} \rangle \sim \frac{F_0(j_i; m_i, \bar{m}_i)}{1 + \sum_i j_i} \quad (2.44)$$

as  $\sum j_i \rightarrow -1$ .  $F_0(j_i; m_i, \bar{m}_i)$  is finite at  $\sum j_i = -1$ .

The poles (2.44) are different in many ways from the external leg poles corresponding to the LSZ reduction, that were discussed in the previous subsections:

- (1) While the LSZ poles are associated with states living near the tip of the cigar, the bulk poles are due to the presence of the semi-infinite linear dilaton direction and are supported infinitely far from the tip. In particular, introducing a physical cutoff which stops  $\phi$  from going to  $+\infty$  regularizes the bulk poles, replacing them by large but finite contributions proportional to the length of the cutoff cigar; the LSZ poles are insensitive to the presence of such a cutoff. The fact that the bulk poles are associated with the region far from the tip of the cigar, where the worldsheet theory simplifies, also allows one to compute the residue of the poles (2.44) using free field techniques, as reviewed in [25].
- (2) The location of the bulk poles depends on the genus of the worldsheet,  $g$ . On the sphere, there is a bulk pole at (2.43), but the same amplitude is finite and dominated by the region near the tip of the cigar for  $g \geq 1$ . The higher genus analog of the condition for the bulk pole (2.43) is

$$\sum_i j_i = g - 1. \quad (2.45)$$

Thus, in different orders in string perturbation theory, one finds different sets of bulk poles; the LSZ poles remain the same.

- (3) The poles (2.44) (and their generalizations described below) depend on all the  $\{j_i\}$  appearing in a correlation function, while the LSZ poles depend only on a single  $j$ .
- (4) The locations of the bulk poles only depend on the  $\phi$ -momentum  $j$ , and not on the other quantum numbers on the cigar  $(m, \bar{m})$ <sup>12</sup>. In contrast, the locations of the LSZ poles (2.22), (2.31) involve in a non-trivial way  $(j; m, \bar{m})$ .

The discussion above can be generalized in two different directions. First, we have only considered the bulk poles due to the leading terms in the large  $\phi$  expansion of the vertex operators (2.20). It turns out that the only other term in this expansion that we need to consider is the leading normalizable term (the last term in (2.20)) – the other terms in the expansion (2.20) do not give any new poles.

Consider, for example, the contribution to the  $n$ -point function (2.10) in which we keep the leading non-normalizable term in the large  $\phi$  expansion of the first  $l$  operators

---

<sup>12</sup> The residues of the poles,  $F_g(j_i; m_i, \bar{m}_i)$ , do in general depend on all the other parameters.

$V_{j_i; m_i, \bar{m}_i}$ ,  $\exp(Qj_i\phi)$ , and the leading normalizable term,  $\exp(-Q(j_i + 1)\phi)$ , for the other  $n - l$ . The consideration after equation (2.42) lead in this case to poles located at

$$\sum_{i=1}^l j_i - \sum_{i=l+1}^n (j_i + 1) = g - 1 . \quad (2.46)$$

Another way of deducing the presence of the poles (2.46) is to use the reflection property satisfied by the observables  $V_{j; m, \bar{m}}$ , (2.7). Applying (2.7) to  $n - l$  of the operators in (2.10) takes (2.45) to (2.46).

The second generalization of the discussion above involves perturbative interactions with the background. The results (2.45), (2.46) were obtained in the leading approximation, where we replace the cigar by an infinite flat cylinder, and are due to interactions among the  $V_{j; m, \bar{m}}$  in the bulk of this cylinder. It is known from studies of CFT in this and related backgrounds that a more general class of bulk amplitudes involves interactions which include both the  $V_{j; m, \bar{m}}$  and the metric. To study such interactions in practice, one expands the metric on the cigar around its large  $\phi$  limit (the metric on a cylinder). This leads to an effective interaction term on the cylinder labeled by  $(\phi, Y)$  (see (2.20)) of the form

$$\mathcal{L}_1 = \lambda \partial Y \bar{\partial} Y e^{-Q\phi} . \quad (2.47)$$

This interaction can be related to the Wakimoto description of  $SL(2, \mathbb{R})$  CFT in terms of free fields (see *e.g.* [22,20]). Bulk interactions involving the metric deformation (2.47) give rise to poles that generalize (2.45), (2.46). Roughly speaking, we can differentiate the correlation functions (2.10)  $n$  times with respect to  $\lambda$ , bringing down  $n$  vertex operators (2.47), and repeat the discussion above. It is easy to see that this leads to poles at the same locations as in (2.45), (2.46), but with  $g$  replaced by  $g + n$ .

Another class of perturbative interactions involves the dual description of the  $SL(2, \mathbb{R})/U(1)$  CFT due to V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [26,27] in terms of Sine-Liouville theory. For our purposes, the main consequence of this duality is that in addition to the metric perturbation (2.47) one should think of the CFT on the cigar as containing in the Lagrangian the Sine-Liouville interaction

$$\mathcal{L}_2 = \mu e^{-\frac{1}{Q}\phi} \cos \left[ \sqrt{\frac{k+2}{2}} (Y - \bar{Y}) \right] . \quad (2.48)$$



This interaction carries one unit of winding, and thus explicitly breaks winding number conservation, in agreement with what one expects from the cigar picture. The precise relation between  $\lambda$  (2.47) and  $\mu$  (2.48) can be found in [20].

The most general bulk pole is obtained by expanding the correlation function (2.10) to order  $n_1$  in  $\lambda$  (2.47) and to order  $n_2$  in  $\mu$  (2.48). It is not difficult to show that the general form of (2.46) is thus

$$\sum_{i=1}^l j_i - \sum_{i=l+1}^n (j_i + 1) = g + n_1 + n_2 \frac{k}{2} - 1, \quad (2.49)$$

where  $n_1$  and  $n_2$  are non-negative integers.

A natural question is whether one can ever confuse LSZ poles in correlation functions with bulk poles. As discussed previously, in general the answer is no, due to the different kinematic structure associated with the two kinds of poles. While the former give poles that depend on the individual  $j_i$ , the latter occur as a function of a particular linear combination of all the  $j_i$  in a correlation function.

One situation in which the two can be confused is in two-point functions, since then there is only one independent  $j$ . We will discuss examples of this in §5.

Another case where one might worry about this issue is the following. Consider the  $n$ -point function (2.10) near the LSZ poles corresponding to the external legs with indices  $2, \dots, n$ . The residue of these LSZ poles is a correlator of  $n - 1$  normalizable operators which behave as  $\phi \rightarrow \infty$  like  $\exp(-Q(j_i + 1)\phi)$ , ( $i = 2, 3, \dots, n$ ), and one non-normalizable operator,  $V_{j_1; m_1, \bar{m}_1}$ . Suppose we now want to take this last operator to an LSZ pole (2.22), (2.31) as well, in order to study the S-matrix (2.13).

A natural question is whether the residue of this pole also receives contributions from a bulk amplitude which satisfies (2.49) with  $l = 1$ . This appears to be possible if the LSZ pole corresponding to  $j_1$  occurs at

$$j_1 = \sum_{i=2}^n (j_i + 1) + n_1 + n_2 \frac{k}{2} - 1 \quad (2.50)$$

for some non-negative integers  $n_1, n_2$ . In particular, (2.50) implies that a necessary condition for a bulk pole to give a contribution with the same kinematic structure as an S-matrix element is

$$j_1 \geq n - 2 + \sum_{i=2}^n j_i. \quad (2.51)$$

For any given S-matrix element (2.13), there are now two possibilities. One is that (2.51) is not satisfied. In this case, there is clearly no bulk contribution to the residue of the LSZ poles.

The second possibility is that (2.51) is satisfied, in which case at least at this level of analysis it looks like a bulk contribution might exist. But in fact, since  $(j_1, j_2, \dots, j_n)$  appear symmetrically in (2.13), we can in that case exchange  $j_1$  with one of the other  $j_i$ , and repeat the analysis. It is easy to show that since all  $j_i$  are larger than  $-1/2$ , for all  $n > 2$  there must exist a choice for which the analog of (2.51) is no longer satisfied. For this choice, it is manifest that there is no bulk contribution that can mix with the LSZ pole.

Thus, we conclude that in  $n \geq 3$  point functions, we can always perform the calculation in a way that makes it manifest that the only contribution to the residue of the poles in (2.13) comes from the S-matrix of the normalizable states localized near the tip of the cigar.

Another natural question is how bulk poles of the sort discussed in this section arise when we compute the correlation functions (2.10) using equation (2.36), by first computing the correlator of  $\Phi_j(x, \bar{x})$  on  $AdS_3$  and then integrating over the  $x_i$ . There are two ways this might happen. One is that the  $AdS_3$  correlation function  $\langle \Phi_{j_1}(x_1) \cdots \Phi_{j_n}(x_n) \rangle$  has such poles even before we integrate over the  $\{x_i\}$ . Examples of such poles were studied in [28,20,24].

The other is a contribution from regions in the integrals over the  $\{x_i\}$  (2.36) in which two or more of them approach each other. A useful way of thinking about these regions is from the point of view of the  $AdS_3/CFT_2$  correspondence. The behavior of correlation functions in the spacetime  $CFT_2$  as some of the  $x_i$  approach each other is mapped by the correspondence to physics that occurs near the boundary of  $AdS_3$ . After modding out by  $U(1)$  to go from  $AdS_3$  to  $SL(2, \mathbb{R})/U(1)$ , the boundary of  $AdS_3$  becomes the region far from the tip of the cigar. Thus, it is natural to expect that singularities of the  $x_i$  integrals that come from these regions correspond to bulk processes on the cigar.

The analysis of the regions  $x_i \rightarrow x_j$  to the integral (2.36) is familiar from studies of string amplitudes, where the role of the  $x_i$  is played by the string worldsheet coordinates. We will present an example of both sources of bulk poles in a particular correlation function in the next subsection.

## 2.5. The three-point function

The discussion of the previous subsections can be made quite explicit for the case of the three-point function (2.10), since the relevant correlation function in  $AdS_3$  is known [18]. In this subsection we will describe how various elements of that discussion manifest themselves in this case. For simplicity we will restrict to pure winding modes,  $m_i = \bar{m}_i$ , with  $m_1 + m_2 + m_3 = 0$ , and (without loss of generality) take  $m_1, m_2 < 0, m_3 > 0$ .

The three-point function (2.36) takes in this case the form

$$\begin{aligned} \langle V_{j_1; m_1, m_1} V_{j_2; m_2, m_2} V_{j_3; m_3, m_3} \rangle &= D(j_1, j_2, j_3; k) \times \\ &\int d^2 x_1 d^2 x_2 d^2 x_3 |x_1|^{2(j_1 + m_1)} |x_2|^{2(j_2 + m_2)} |x_3|^{2(j_3 + m_3)} \times \\ &|x_1 - x_2|^{2(j_3 - j_1 - j_2 - 1)} |x_1 - x_3|^{2(j_2 - j_1 - j_3 - 1)} |x_2 - x_3|^{2(j_1 - j_2 - j_3 - 1)}, \end{aligned} \quad (2.52)$$

where

$$\begin{aligned} D(j_1, j_2, j_3; k) &= \frac{k}{2\pi^3} \nu(k)^{j_1 + j_2 + j_3 + 1} \times \\ &\frac{G(-j_1 - j_2 - j_3 - 2) G(j_3 - j_1 - j_2 - 1) G(j_2 - j_1 - j_3 - 1) G(j_1 - j_2 - j_3 - 1)}{G(-1) G(-2j_1 - 1) G(-2j_2 - 1) G(-2j_3 - 1)}. \end{aligned} \quad (2.53)$$

$\nu(k)$  is given by (2.8);  $G(x)$  is a known function, whose analytic properties are described in appendix A. The factor  $C_{123}$  in (2.36) can be omitted in this case.

To study the LSZ poles, one is interested in the contribution to the integrals (2.52) from  $x_i \rightarrow 0, \infty$ . As in the general discussion leading to equation (2.39), the leading divergence as  $x_1 \rightarrow 0, x_3 \rightarrow \infty$  gives rise to a pole at  $j_1 = -m_1 - 1, j_3 = m_3 - 1$ . Expanding (2.52) near this pole we find that it behaves as

$$\langle V_{j_1; m_1, m_1} V_{j_2; m_2, m_2} V_{j_3; m_3, m_3} \rangle \simeq \frac{\pi^2 D(j_1, j_2, j_3; k)}{(m_1 + j_1 + 1)(m_3 - j_3 - 1)} \int \frac{d^2 x_2}{x_2^2}. \quad (2.54)$$

The infinite factor  $\int \frac{d^2 x_2}{x_2^2}$  imposes the momentum and winding conservation conditions (2.34) [14]. In the decomposition of  $SL(2, \mathbb{R})$  as  $\frac{SL(2, \mathbb{R})}{U(1)} \times U(1)$ , it belongs to the  $U(1)$  part. Therefore, it appears uniformly in all correlators (as we will exhibit explicitly below) and should be ignored.

At first sight it seems that since the  $x_2$  integral only imposes momentum and winding conservation, we are missing the LSZ poles of the correlation function (2.52) that involve  $j_2$ . In fact, this is not the case – these poles come from the “structure function”  $D(j_1, j_2, j_3; k)$ .

To see that, recall that since  $m_2$  is negative, we expect to find poles when  $m_2 + j_2$  is a negative integer. When computing the residue of the poles in (2.54) we can set  $m_1 + j_1 + 1 = m_3 - j_3 - 1 = 0$ . Hence,

$$j_2 + m_2 = j_2 - m_1 - m_3 = j_2 + j_1 - j_3. \quad (2.55)$$

The function  $D(j_1, j_2, j_3; k)$  (2.53) has a factor,  $G(j_3 - j_1 - j_2 - 1)$ , which has a pole whenever its argument,  $j_3 - j_1 - j_2 - 1$ , is a non-negative integer. These poles correspond, via (2.55), to poles as a function of  $m_2 + j_2$  which occur at precisely the right places to correspond to contributions to the S-matrix.

While the pole in  $j_2$  appears in a different way than those in  $j_1$  and  $j_3$ , it is an LSZ pole due to normalizable states and does not receive a bulk contribution. This follows from the general discussion in the previous subsection. Indeed, suppose we wanted to compute the bulk contribution to the residue of the LSZ poles in  $j_1$  and  $j_3$  (2.54). Since the residue of the poles involves normalizable vertex operators (see (2.20)), the leading large  $\phi$  behavior of the correlator computing this residue is  $\langle e^{-Q(j_1+1)\phi} e^{Qj_2\phi} e^{-Q(j_3+1)\phi} \rangle$ . The total power of  $e^{-Q\phi}$  coming from the vertex operators is  $j_1 + j_3 + 2 - j_2$ . Near a pole at  $m_2 + j_2 = -n$  (where  $n$  is a positive integer), we can use equation (2.55) to rewrite this as  $j_1 + j_3 - j_2 + 2 = 2j_1 + n + 2$ . Thus, the  $\phi$  zero mode integral goes at large  $\phi$  as  $\int d\phi_0 e^{-Q(2j_1+1+n)\phi_0}$ . Using the fact that  $2j_1 + 1 \geq 0$  and  $n \geq 1$ , we see that the  $\phi_0$  integral rapidly converges at infinity, and therefore there cannot be a bulk contribution to the pole at  $m_2 + j_2 = -n$ . Insertions of the interactions (2.47), (2.48) suppress the  $\phi_0$  integral even more. Hence, we conclude that this pole is entirely due to the contribution of an on-shell normalizable state – it is an LSZ pole.

The following question might seem puzzling at this point. The factor  $G(j_3 - j_1 - j_2 - 1)$  from which we got the  $j_2$  LSZ pole is an overall factor in the three-point function (2.52). It gives rise to poles of the three-point function whenever  $j_3 - j_1 - j_2 - 1$  is a non-negative integer, irrespective of the values of the  $\{m_i\}$ . Hence, the analysis of the previous subsections leads one to believe that these poles can be interpreted as bulk poles. As we will see next, this is indeed the case for generic values of the  $j_i$ , but it does not contradict the discussion above, which interpreted the same poles as LSZ poles.

Looking back at equation (2.49) we see that in order to interpret the poles of the factor  $G(j_3 - j_1 - j_2 - 1)$  as bulk poles we have to focus on the contribution to the three-point function in which we take the leading non-normalizable contribution from  $V_{j_3}$ , and

the leading normalizable terms from  $V_{j_1}, V_{j_2}$  (see (2.20)). For generic  $j_1$  and  $j_3$ , the poles which appear when  $j_3 - j_1 - j_2 = n = 1, 2, \dots$  are indeed interpreted as bulk poles.

To compare to the preceding analysis, in which the same poles were interpreted as LSZ poles, we need to take the external legs on-shell. In the discussion around equation (2.54) this was done by first taking  $j_1 \rightarrow -m_1 - 1$  and  $j_3 \rightarrow m_3 - 1$  and then studying the origin of the third pole of the three-point function (2.52) associated with  $j_2$ .

In the present way of computing the amplitude it is convenient to first take the external legs  $j_1$  and  $j_2$  on-shell and then study the singularity structure as a function of  $j_3$ . Indeed, the residue of the poles at (say)  $j_1 = -m_1 - 1$ ,  $j_2 = -m_2 - 1$  is proportional to the three-point function of two normalizable operators, corresponding to  $j_1$  and  $j_2$ , and a third operator  $V_{j_3}$ , which has both normalizable and non-normalizable parts. Thus, in this way of calculating the correlator, one would say that there are two contributions to the pole at  $j_3 = m_3 - 1$ . One is an LSZ pole, whose residue is the three-point function of normalizable operators. The other is a bulk pole; its residue is the three-point function of two normalizable ( $j_1$  and  $j_2$ ) and one non-normalizable ( $j_3$ ) operators.

This situation is an example of the general discussion around equation (2.50). If we first go to the LSZ poles in  $j_1$  and  $j_2$ , the LSZ pole in  $j_3$  seems to mix with a bulk pole. As argued there, we can always perform the calculation in such a way that the bulk contribution is manifestly absent. Here, one way to do that is to first go to the LSZ poles in  $j_1$  and  $j_3$ , as we have done in (2.54). In that way of doing the calculation it is clear that the bulk contribution to the last pole (in  $j_2$ ) vanishes.

Thus, we see that the same pole in the three-point function can sometimes be interpreted as a bulk pole, and sometimes as an LSZ pole. There is clearly more to be said on this subject; in particular, it would be interesting to understand how the apparent bulk contributions to the three-point function cancel near the LSZ poles in the second way of calculating it. We will not pursue this and other similar questions here.

So far we have mainly studied the way that LSZ poles appear in the three-point function (2.52). We next discuss briefly the different bulk poles exhibited by these amplitudes<sup>13</sup>. We have already seen one source of such poles, corresponding to singularities of

---

<sup>13</sup> We will think of  $k$  as large, and study the bulk poles at  $j_i$  of order one. There are additional poles at which some or all of the  $j_i$  must be of order  $k$ . They can easily be studied in the same way. We will also assume that  $k$  is irrational. For rational  $k$  (the case of interest in applications) there are some further issues that need to be discussed.

the function  $G(j_3 - j_1 - j_2 - 1)$  in the  $AdS_3$  form factor (2.53). Clearly, there are similar poles that are due to the other two factors of  $G$  in (2.53) that are obtained by permutations of  $(j_1, j_2, j_3)$ .

The remaining factor in the numerator of (2.53),  $G(-j_1 - j_2 - j_3 - 2)$ , has poles when  $j_1 + j_2 + j_3 + 2 = 0, -1, -2, \dots$ . In the general analysis (2.49) such poles can in principle arise in a contribution with  $l = 0$ . However, due to the constraint  $j_i > -1/2$ , these poles are actually out of the physical range.

As mentioned in the previous subsection, an additional source of bulk poles is the regions in the integrals over  $x_i$  where two or more of them approach each other. Consider, for example, the contribution to the three-point function (2.52) from the region  $x_1 \rightarrow x_2$ . This region leads to poles located at

$$j_1 + j_2 - j_3 = 0, 1, 2, \dots \quad (2.56)$$

The residues of these poles are easy to compute using (2.52). Comparing to (2.49) we see that these poles are bulk poles occurring in the contribution to the three-point function where we take the leading non-normalizable terms in  $j_1$  and  $j_2$ , and the leading normalizable term in  $j_3$ .

Note also that the poles (2.56) do not overlap those due to the factor  $G(j_3 - j_1 - j_2 - 1)$  in the  $AdS_3$  structure constant. This is a necessary condition for the interpretation of these poles as bulk contributions, since the integral over the zero mode of  $\phi$  can only give single poles, and not double poles.

Similarly, one can study the poles of (2.52) that come from the region where all three of the  $x_i$  approach each other. Changing variables from  $(x_2, x_3)$  to  $(\epsilon, y)$  where  $x_2 - x_1 = \epsilon$ ,  $x_3 - x_1 = \epsilon y$ , and studying the contribution of the region  $\epsilon \rightarrow 0$  to (2.52), it is not difficult to see that this region gives poles when

$$j_1 + j_2 + j_3 = -1, 0, 1, 2, 3, \dots \quad (2.57)$$

These poles are due to the leading term in the three-point function (2.52), in which we replace all three vertex operators by their leading, non-normalizable contributions as  $\phi \rightarrow \infty$ . Indeed, (2.57) is in agreement with (2.49), for the relevant case  $l = 0$ .

Many elements of the discussion above generalize to  $n > 3$  point functions. The fact that we can see only  $n - 1$  of the  $n$  LSZ poles in (2.13) by studying the region  $x_i \rightarrow 0, \infty$

follows from the properties of the  $n$ -point function of the  $\Phi_{j_i}(x_i)$  (2.36) under rescaling of the  $x_i$ . Using the fact that

$$\langle \Phi_{j_1}(\lambda x_1) \cdots \Phi_{j_n}(\lambda x_n) \rangle = |\lambda|^{-2 \sum_i (j_i+1)} \langle \Phi_{j_1}(x_1) \cdots \Phi_{j_n}(x_n) \rangle, \quad (2.58)$$

and changing variables from  $(x_1, \dots, x_n)$  to  $(y_1, \dots, y_{n-1}, x_n)$ , with  $y_i = x_i/x_n$ , one finds that

$$\begin{aligned} \prod_{i=1}^n \int d^2 x_i x_i^{j_i+m_i} \bar{x}_i^{j_i+\bar{m}_i} \langle \Phi_{j_1}(x_1) \cdots \Phi_{j_n}(x_n) \rangle = \\ \prod_{i=1}^{n-1} \int d^2 y_i y_i^{j_i+m_i} \bar{y}_i^{j_i+\bar{m}_i} \langle \Phi_{j_1}(y_1) \cdots \Phi_{j_{n-1}}(y_{n-1}) \Phi_{j_n}(1) \rangle \int d^2 x_n x_n^{\sum_i m_i - 1} \bar{x}_n^{\sum_i \bar{m}_i - 1}. \end{aligned} \quad (2.59)$$

Thus, one of the  $n$  integrals over  $x_i$  (which in (2.59) has been chosen to be the one over  $x_n$ ) can be thought of as imposing momentum and winding conservation, just like we have found for the three-point function (see (2.54)). Presumably, the last LSZ pole arises in all  $n$ -point functions from the behavior of the unintegrated  $n$ -point function on  $AdS_3$ , like we saw it does in the three-point function. It would be interesting to understand this in more detail.

The fact that the last pole appears in this case at the same value of  $j_n$  as a bulk pole is general as well. Suppose, for example, that  $m_1, \dots, m_l > 0$  and  $m_{l+1}, \dots, m_n < 0$ . Take the first  $n-1$  external legs to LSZ poles with  $j_i = m_i - 1 - s_i$  for  $i = 1, \dots, l$  and  $j_i = -m_i - 1 - s_i$  for  $i = l+1, \dots, n-1$ , with  $s_i \in \mathbf{Z}_+$ . The location of the last LSZ pole, which is expected to occur at  $j_n = -m_n - 1 - s_n$ , can be rewritten using momentum and winding conservation (2.34) as

$$\sum_{i=1}^l j_i - \sum_{i=l+1}^n j_i = n - 1 - 2l + \sum_{i=l+1}^n s_i - \sum_{i=1}^l s_i. \quad (2.60)$$

This has the general form (2.49) and hence corresponds to a position of a bulk pole. However, as discussed in the previous subsection, one can always perform the calculation in such a way that it is manifest that the S-matrix (2.13) does not receive contributions from bulk poles.

## 2.6. Winding non-conserving correlators

So far in this section we focused on  $SL(2, \mathbb{R})/U(1)$  correlation functions (2.10) that preserve both momentum and winding (2.34). As mentioned in the beginning of §2.3, winding conservation can actually be violated on the cigar. In this subsection we briefly comment on the generalization of the analysis above to winding violating correlation functions, leaving a more detailed discussion to future work.

A nice way to study winding number violating amplitudes, which is reminiscent of an analogous construction in the  $SU(2)/U(1)$  CFT, was proposed by [26]. Consider the  $AdS_3$  operator<sup>14</sup>  $\Phi_{j;m,\bar{m}}$  (2.17), with  $m = \bar{m} = -j = \frac{k+2}{2}$ . This operator belongs to a degenerate representation of the  $SL(2, \mathbb{R})$  affine Lie algebra (see *e.g.* §4.2 in [20]). Comparing to (2.5), we see that this operator carries one unit of winding number,  $w$ . At the same time, (2.4) implies that the  $SL(2, \mathbb{R})/U(1)$  part of this operator has dimension  $\Delta = \bar{\Delta} = 0$ .

The basic idea of [26] is that this operator should be interpreted as a product of an operator in the  $U(1)$  CFT, and the identity operator in the coset. Thus, to study correlation functions in the coset, (2.10), that violate winding number by  $n$  units, we can start with the  $AdS_3$  correlation function (2.36) with  $n$  extra insertions of the operator  $\Phi_{-\frac{k+2}{2}; \frac{k+2}{2}, \frac{k+2}{2}}$  (or its complex conjugate), and compute the resulting  $AdS_3$  correlation function, which satisfies (2.34). Stripping off the  $U(1)$  part (which can be easily computed) we find the winding number violating  $SL(2, \mathbb{R})/U(1)$  correlation function we are after.

Another (related) point that can be made about winding number violating amplitudes on the cigar is the following. Suppose we want to compute the residue of an LSZ pole in one or more of the external legs (see *e.g.* (2.12)). Then, at least one of the operators in (2.10) is normalizable, *e.g.* for  $m < -1/2$  it could be:

$$V_{-m-1;m,m}^{(\text{norm})} = \lim_{j+m+1 \rightarrow 0} (j+m+1) V_{j;m,m}. \quad (2.61)$$

Then, a reflection symmetry<sup>15</sup> relating naively different states in  $SL(2, \mathbb{R})/U(1)$  implies that, in a suitable normalization of the operators,

$$V_{-m-1;m,m}^{(\text{norm})} = V_{\frac{k}{2}+m; m+\frac{k+2}{2}, m+\frac{k+2}{2}}^{(\text{norm})}. \quad (2.62)$$

---

<sup>14</sup> Note that the reflection formula (2.7) naively relates this operator to an operator with  $j = \frac{k}{2}$ ,  $m = \bar{m} = \frac{k+2}{2}$ , which violates the bound mentioned at the end of §2.3. This is not necessarily a problem since the reflection relation (2.7) does not obviously apply when one of the operators it relates violates this bound; it originates in CFT on Euclidean  $AdS_3$ , where such operators are not expected to exist.

<sup>15</sup> This is not to be confused with the different reflection property (2.7).



In other words, different non-normalizable operators, in this case the operators  $V_{j;m,m}$  and  $V_{\frac{k-2}{2}-j;m+\frac{k+2}{2},m+\frac{k+2}{2}}$ , can create the same state from the vacuum. This phenomenon was noticed in [29,30], and given this interpretation in [3]. In field theory it is not surprising that the same state may be created by more than one operator, and it was verified in [3] in a specific example that (2.62) is consistent with the low-energy field theory content of these operators.

Since the winding number of the right and left hand sides of (2.62) differs by one unit, we can use it to relate correlation functions (involving normalizable operators) that violate winding number to those that preserve it. Examples of winding-non-conserving correlators will be given in §5.6.

### 3. Superstring theory on $SL(2, \mathbb{R})/U(1)$

Bosonic string theory on spacetimes of the form (1.4) is in general IR unstable (for  $d > 0$  or non-trivial  $\tilde{\mathcal{M}}$ ), a fact that typically manifests itself in the presence of tachyons in the spectrum of normalizable states, and IR divergences in loop amplitudes. To avoid these instabilities we turn in this section to the superstring. This is the case which is relevant for the applications of our formalism to the decoupled theories of fivebranes and Calabi-Yau singularities in type II string theory, which we will discuss below.

There are two steps involved in generalizing the discussion of §2 to the superstring. First, we need to supersymmetrize the worldsheet theory and enlarge the gauge principle from  $\mathcal{N} = 0$  to  $\mathcal{N} = 1$  supergravity. This leads to type 0 string theory and in itself does not solve the IR problems of the bosonic string. To achieve that, one needs to also perform a chiral GSO projection [31,32].

The supersymmetric level  $k$   $SL(2, \mathbb{R})/U(1)$  CFT can be constructed as follows. We start with a bosonic  $SL(2, \mathbb{R})$  WZW model with central charge  $c = 3(k+2)/k$ , as in (2.1), and add to it three free fermions  $\lambda^a$ ,  $a = 3, \pm$  (and their anti-chiral analogs). Associated with the fermions  $\lambda^a$  is an  $SL(2, \mathbb{R})$  current algebra of level  $(-2)$ . The full theory is  $\mathcal{N} = 1$  superconformal; we review the structure of its superconformal algebra in appendix A. The fermions  $\lambda^a$  are bottom components of superfields whose top components are the total  $SL(2, \mathbb{R})$  currents  $J_a^{(\text{total})}$ . The level of this current algebra is  $k + 2 + (-2) = k$ , where the two contributions come from the bosonic  $SL(2, \mathbb{R})$  WZW model and from the fermions.

To describe the coset, we would like to gauge the  $U(1)$  superfield whose bottom component is  $\lambda_3$  and whose top component is the total  $U(1)$  current  $J_3^{(\text{total})} = \{G_{-\frac{1}{2}}, \lambda_3\}$ . This leads to a SCFT with central charge

$$c_{sl} = \frac{3(k+2)}{k} + \frac{3}{2} - \frac{3}{2} = \frac{3(k+2)}{k}, \quad (3.1)$$

where the  $+\frac{3}{2}$  is the contribution of the fermions  $\lambda^a$ , and the  $-\frac{3}{2}$  is due to the gauging. While the underlying CFT on  $SL(2, \mathbb{R})$  is  $\mathcal{N} = 1$  superconformal, the coset  $SL(2, \mathbb{R})/U(1)$  is actually invariant under an  $\mathcal{N} = 2$  superconformal symmetry. The generators of the  $\mathcal{N} = 2$  algebra are given in appendix A.

The requirement that (1.4) corresponds to a solution of the classical equations of motion of the superstring leads to the constraint (compare to (2.2))

$$\frac{3}{2}d + c_{sl} + c_{\mathcal{M}} = 15. \quad (3.2)$$

To study perturbative string theory on (1.4) we need to construct vertex operators on  $SL(2, \mathbb{R})/U(1)$ . This can be done as in section 2, by studying vertex operators in the underlying SCFT on  $AdS_3$ , and then removing from them the  $U(1)$  part. We will describe this here for the Neveu-Schwarz (NS) sector. The analysis of the Ramond sector is similar; for a recent discussion, see [3].

We can again take as a starting point the  $SL(2, \mathbb{R})$  vertex operators  $\Phi_{j;m,\bar{m}}$  (2.17). While these operators do not contain the worldsheet fermions  $\lambda^a$ , removing their  $U(1)$  part does *not* lead to the operators  $V_{j;m,\bar{m}}$  that we encountered in section 2. The reason is that the  $U(1)$  current  $J_3^{(\text{total})}$  that is being gauged includes contributions from both the bosonic WZW model and from the fermions. Removing the  $U(1)$  part of the operators  $\Phi_{j;m,\bar{m}}$  in the superstring gives operators that we will denote by  $V_{j;m,\bar{m}}^{(sl,susy)}$ , whose dimension is (compare to (2.4))

$$\begin{aligned} \Delta_{j;m} &= \frac{m^2 - j(j+1)}{k}, \\ \bar{\Delta}_{j;\bar{m}} &= \frac{\bar{m}^2 - j(j+1)}{k}. \end{aligned} \quad (3.3)$$

The operators  $V_{j;m,\bar{m}}^{(sl,susy)}$  are in fact primaries of the full  $\mathcal{N} = 2$  superconformal symmetry. Their R-charges are given by

$$\begin{aligned} R_m &= \frac{2m}{k}, \\ \bar{R}_{\bar{m}} &= \frac{2\bar{m}}{k}. \end{aligned} \quad (3.4)$$

Far from the tip, the cigar looks like a semi-infinite cylinder. As in the bosonic case, we will parametrize the semi-infinite direction along this cylinder by  $\phi$ . The direction around the cylinder,  $Y$ , now lives on a circle of radius  $\sqrt{2k}$ . The momentum and winding in the  $Y$  direction,  $(n, w)$ , are related to  $(m, \bar{m})$  in (3.3) in a way similar to (2.5):

$$\begin{aligned} m &= \frac{1}{2}(wk + n), \\ \bar{m} &= \frac{1}{2}(wk - n). \end{aligned} \tag{3.5}$$

The vertex operator  $V_{j;m,\bar{m}}^{(sl,susy)}$  has the asymptotic form

$$V_{j;m,\bar{m}}^{(sl,susy)} \simeq \frac{e^{iQ(mY - \bar{m}\bar{Y})}}{2j+1} \left[ e^{Qj\phi} + R(j, m, \bar{m}; k) e^{-Q(j+1)\phi} + \dots \right], \tag{3.6}$$

where the reflection coefficient  $R(j, m, \bar{m}; k)$  is given by (2.7), and  $Q$  is related to  $k$  by (1.5).

Other observables on the cigar can be obtained by acting on  $V_{j;m,\bar{m}}^{(sl,susy)}$  by the  $\mathcal{N} = 2$  superconformal symmetry generators. For example, at the first excited level we find the operators  $[G_{-\frac{1}{2}}^{\pm}, V_{j;m,\bar{m}}^{(sl,susy)}]$  (and similarly for the other worldsheet chirality). Far from the tip of the cigar the theory simplifies into that of two free superfields  $(\phi, \psi_{\phi})$  and  $(Y, \psi_Y)$ , and these excited operators can be written in terms of the  $\mathcal{N} = 2$  descendants  $\psi_{\phi} e^{ipY + \beta\phi}$  and  $\psi_Y e^{ipY + \beta\phi}$ . At higher excited levels the situation is similar.

A class of physical observables in superstring theory on (1.4) can be constructed as follows. Let  $\mathcal{W}$  be an  $\mathcal{N} = 1$  superconformal primary on  $\tilde{\mathcal{M}}$  with scaling dimension  $(\Delta_L, \Delta_R)$ . An (NS,NS) sector physical operator can be formed by “dressing”  $\mathcal{W}$  as in (2.3):

$$\mathcal{O}_{\mathcal{W}}(p) = e^{-\varphi - \bar{\varphi}} \mathcal{W} V_{j;m,\bar{m}}^{(sl,susy)} e^{ip \cdot x}. \tag{3.7}$$

Here,  $(\varphi, \bar{\varphi})$  are bosonized superconformal ghosts, and the factor  $e^{-\varphi - \bar{\varphi}}$  indicates that the vertex operator (3.7) is written in the  $(-1, -1)$  picture. The physical state condition for (3.7) is

$$\frac{1}{2}p_{\mu}^2 + \Delta_{j;m} + \Delta_L = \frac{1}{2}p_{\mu}^2 + \bar{\Delta}_{j;\bar{m}} + \Delta_R = \frac{1}{2}. \tag{3.8}$$

As mentioned earlier, in order to study superstring propagation on (1.4) we need to perform a chiral GSO projection. A sufficient condition for being able to do that is that the full background (1.4) is  $\mathcal{N} = 2$  superconformal. This is the case, *e.g.*, if  $d$  is even and the conformal theory on  $\tilde{\mathcal{M}}$  is  $\mathcal{N} = 2$  superconformal. In that case, the chiral GSO projection

amounts to the requirement that the total R-charge of the vertex operator (3.7) must be an odd integer. The GSO projection thus acts as an orbifold projection on the background (1.4). We will see below that an important consequence of this orbifold is the appearance of states with non-integer winding  $w$  (3.5) in the twisted sectors.

Since the observables in the superconformal field theory on the cigar can be lifted to those on (supersymmetric)  $SL(2, \mathbb{R})$ , we can use the results of the analysis of §2 to study the analytic structure of their correlation functions. There are again LSZ poles at the locations (2.22), (2.31) and bulk poles at the locations (2.49). The Wakimoto and Sine-Liouville perturbations (2.47), (2.48) have a slightly different form in this case (see *e.g.* [20]) but this does not alter the analysis leading to (2.49). For example, the analog of the Sine-Liouville perturbation (2.48) is in this case the  $\mathcal{N} = 2$  Liouville perturbation,

$$\mathcal{L}_2 = \mu G_{-\frac{1}{2}}^- \bar{G}_{-\frac{1}{2}}^- e^{-\frac{1}{Q}(\phi - i(Y - \bar{Y}))} + \text{c.c.} \quad (3.9)$$

Here  $\mu$  is the (generally complex)  $\mathcal{N} = 2$  Liouville coupling.

In the next sections we will use the general formalism of §2 and this section to study the dynamics associated with  $NS5$ -branes and singularities of Calabi-Yau manifolds in superstring theory.

## 4. Six dimensional $\mathcal{N} = (1, 1)$ supersymmetric LST

### 4.1. Review

An important application of the formalism developed above is to the dynamics of Neveu-Schwarz fivebranes (or, equivalently, ADE singularities of  $K3$  surfaces). The main example which we will discuss in detail in this paper is a system of  $k$  parallel  $NS5$ -branes, extended in the directions  $(x^0, x^1, \dots, x^5)$ , in type IIB string theory. At low energies the dynamics of this system includes a supersymmetric Yang-Mills (SYM) theory with  $\mathcal{N} = (1, 1)$  supersymmetry in six dimensions (*i.e.* sixteen supercharges), and gauge group<sup>16</sup>  $SU(k)$  (there is also a decoupled  $U(1)$  gauge field which will play no role in this paper). The full theory on the fivebranes can be thought of as a UV completion of this non-renormalizable gauge theory. We will see later that the Green functions of this theory exhibit some unexpected features at low energies.

---

<sup>16</sup> There are also versions of the construction with other simply laced (ADE) gauge groups. We will not discuss them here.

On general grounds, one expects the decoupled theory on the fivebranes to be holographically related to string theory in the near-horizon geometry of the branes. As discussed in section 1, the near-horizon geometry of  $k$  coincident fivebranes is [10]

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k, \quad (4.1)$$

where  $\phi$  is related to the radial direction away from the branes, and the supersymmetric  $SU(2)_k$  WZW model describes the angular three-sphere. The  $SU(2)_L \times SU(2)_R$  symmetry of the WZW model is identified with the  $SO(4)$  rotation symmetry about the fivebranes, and gives rise to an R-symmetry of the LST. The dilaton depends linearly on  $\phi$ , as in (1.1); the slope  $Q$  is related to the number of fivebranes  $k$  by the relation (1.5).

The six dimensional LST has a moduli space  $\mathbb{R}^{4k}/S_k$ , corresponding to the positions of the  $k$   $NS5$ -branes in the transverse  $\mathbb{R}^4$ . To avoid the strong coupling singularity of (4.1) it is convenient to separate the branes in the transverse  $\mathbb{R}^4$ . Parametrizing this  $\mathbb{R}^4$  by the complex coordinates

$$\begin{aligned} a &= x^6 + ix^7, \\ b &= x^8 + ix^9, \end{aligned} \quad (4.2)$$

we will consider the point in moduli space at which the  $l$ 'th fivebrane ( $l = 1, \dots, k$ ) is located at

$$(a, b) = r_0(0, e^{\frac{2\pi i l}{k}}). \quad (4.3)$$

This corresponds to a configuration in which the fivebranes are evenly spaced on a circle of radius  $r_0$  in the  $(x^8, x^9)$  plane. The  $SO(4)$  rotation symmetry around the fivebranes is broken in the background (4.3) to  $SO(2)_{67} \times \mathbf{Z}_k$ . Thus, this is a rather symmetric point in moduli space – generically, the rotation symmetry is broken completely. It should be possible to generalize the discussion below to generic points in the moduli space; we will briefly comment on this problem below (in section 11.2).

In the low energy gauge theory on the fivebranes, the coordinates  $x^i$ ,  $i = 6, 7, 8, 9$ , or  $a$  and  $b$  (4.2), are promoted to scalar fields in the adjoint representation of  $SU(k)$ , which we will denote by  $X^i$ ,  $A$  and  $B$ , respectively. The displacement of the fivebranes (4.3) corresponds to giving an expectation value to  $B$ , of the form

$$\langle A \rangle = 0; \quad \langle B \rangle = M_W M_s \text{diag}(e^{\frac{2\pi i}{k}}, e^{\frac{4\pi i}{k}}, \dots, e^{\frac{2\pi i(k-1)}{k}}, 1). \quad (4.4)$$

The expectation value (4.4) breaks the gauge symmetry from  $SU(k)$  to  $U(1)^{k-1}$ . The off-diagonal components of the matrices  $A$ ,  $B$ , etc, which correspond to D-strings stretched between the fivebranes, get a mass proportional to

$$M_W = r_0 M_s^2 / g_s^{(\text{far})}, \quad (4.5)$$

where  $g_s^{(\text{far})}$  is the value of the string coupling far from the fivebranes.

To decouple the fivebranes from the bulk, we consider the double scaling limit [13]

$$r_0, g_s^{(\text{far})} \rightarrow 0; \quad \frac{r_0}{g_s^{(\text{far})} l_s} = \frac{M_W}{M_s} = \text{fixed}. \quad (4.6)$$

In this limit, fundamental strings propagating in the vicinity of the fivebranes see the “near-horizon geometry”

$$\mathbb{R}^{5,1} \times \left( \frac{SL(2, \mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)} \right) / \mathbf{Z}_k. \quad (4.7)$$

The geometry (4.7) is a cut-off version of (4.1); it reduces to (4.1) far from the tip of the cigar. It was mentioned above that the brane configuration (4.3) spontaneously breaks the rotation symmetry about the fivebranes from  $SO(4)$  to  $SO(2) \times \mathbf{Z}_k$ . In the background (4.7) this corresponds to the fact that while the asymptotic large  $\phi$  geometry has an  $SU(2)_L \times SU(2)_R$  symmetry, the full background (4.7) is only invariant under an  $SO(2) \times \mathbf{Z}_k$  subgroup. The  $SO(2)$  corresponds to translations of  $Y$ , the angular coordinate around the cigar. The  $\mathbf{Z}_k$  is a “quantum symmetry” of the  $\mathbf{Z}_k$  orbifold in (4.7). Note that, as usual in holography, a spontaneously broken symmetry in the gauge theory corresponds in the bulk description to a symmetry that is preserved near the boundary at  $\phi = \infty$ , but is broken at finite  $\phi$ .

The string coupling at the tip of the cigar in (4.7) is:

$$g_s^{(\text{tip})} \simeq M_s / M_W. \quad (4.8)$$

In order for perturbative string theory in the background (4.7) to be reliable,  $g_s^{(\text{tip})}$  has to be small. Indeed, the non-perturbative states corresponding to D-strings stretched between the fivebranes correspond in the geometry (4.7) to D-branes localized near the tip of the cigar. Their mass is thus proportional to  $M_s / g_s^{(\text{tip})} = M_W$ , and requiring that they are much heavier than perturbative string excitations leads to the condition

$$M_W \gg M_s. \quad (4.9)$$

Our computations below will be done in this limit, which was referred to in [13,14] as Double Scaled Little String Theory (DSLST).

The  $\mathbf{Z}_k$  orbifold acts on the SCFTs on the cigar and on  $SU(2)/U(1)$  (which is an  $\mathcal{N} = 2$  minimal model) in (4.7) as follows. The compact coordinate on the cigar,  $Y$ , lives on a circle whose asymptotic radius is  $R = \sqrt{2k}$  (see §3). The orbifold (4.7) acts as a translation,  $Y \rightarrow Y + \frac{2\pi R}{k}$  (or, alternatively, a rotation of the cigar by the angle  $2\pi/k$ ). Clearly this generates a  $\mathbf{Z}_k$  action on the cigar. The  $\mathcal{N} = 2$  minimal model  $SU(2)_k/U(1)$  has a discrete  $\mathbf{Z}_k$  symmetry, and it is the product of this symmetry with the  $Y$ -translation mentioned above that is gauged in (4.7). One important effect of orbifolding by  $\mathbf{Z}_k$  is that the twisted sectors contain states with fractional winding around the cigar,  $w \in \mathbf{Z}/k$ . The fractional part of  $w$  is the  $\mathbf{Z}_k$  charge.

Another way of presenting the double scaling limit (4.6) corresponds to the description of the near horizon geometry of the separated fivebranes in terms of  $\mathcal{N} = 2$  Liouville theory (3.9). The radius of the circle on which the fivebranes are placed is related to the  $\mathcal{N} = 2$  Liouville coupling via the relation [13]  $r_0/l_s = |\mu|^{\frac{1}{k}}$ . Thus, we see that  $M_W$  scales like

$$M_W \sim |\mu|^{\frac{1}{k}}. \quad (4.10)$$

This relation will be used below for determining the dependence of amplitudes on  $M_W$ , via KPZ scaling.

Since the DSLST background (4.7) is a special case of (1.4), we can use the results of §2, §3 to analyze it. In particular, we can construct physical vertex operators as in §3, and study the analytic structure of Green functions using the results of §2. The main focus of our discussion will be on the behavior of these Green functions at low energies, *i.e.* in situations where all the kinematic invariants  $p_i \cdot p_j$  (including those with  $i = j$ ) are much smaller than  $M_s^2$ .

The low energy limit of the fivebrane theory at the point (4.4) on its moduli space is expected to be a  $U(1)^{k-1}$  gauge theory with sixteen supercharges. Thus, one might expect the physical vertex operators in string theory on (4.7) to reduce at low energies to local operators in that gauge theory, and the correlation functions of these vertex operators to reduce to off-shell Green functions in the gauge theory. Of course, since the gauge theory is non-renormalizable, we expect its Lagrangian to include high dimension operators, such as  $\text{tr}(F^{2n})$  with  $n > 1$  (appropriately supersymmetrized).

We will see that the actual situation is more subtle. The spectrum of massless states is indeed in one to one correspondence with states in the  $U(1)^{k-1}$  gauge theory with sixteen

supercharges. The residues of LSZ poles as a function of the external momenta agree with low energy expectations as well, *i.e.* they can be obtained from a gauge-invariant effective action which has a power series expansion in local gauge-invariant fields, when we assume that the string theory vertex operators in the background (4.7) reduce at low energies to particular gauge-invariant operators in the field theory.

However, as we saw in §2, the off-shell Green functions obtained from string theory have some additional singularities associated with bulk dynamics. We will see that some of these singularities influence the low energy behavior of correlation functions and they cannot be described by an effective action written purely in terms of the low energy interpolating fields. Thus, the statement that the low energy theory on  $k$   $NS5$ -brane in type IIB string theory along its Coulomb branch is a  $U(1)^{k-1}$  gauge theory is not quite accurate off-shell, even at arbitrarily large distances.

To make the above discussion more concrete, we will study below a particular class of string theory observables, and try to match their correlation functions to those of the corresponding operators in the low energy gauge theory.

The following non-normalizable vertex operators in the fivebrane background (4.1) were identified (at low energies) with operators in the low energy gauge theory in [9,3]:

$$e^{-\varphi-\bar{\varphi}}(\psi\bar{\psi}\Phi_j^{(su)})_{j+1;m,\bar{m}}e^{Q\tilde{j}\phi}e^{ip\cdot x}\leftrightarrow\widetilde{\text{tr}}(X^{i_1}X^{i_2}\dots X^{i_{2j+2}}). \quad (4.11)$$

The notation in (4.11) is as follows.  $\psi^a$ , with  $a = 3, \pm$ , are three free fermions which transform in the adjoint of  $SU(2)_L$  (and similarly for  $\bar{\psi}^{\bar{a}}$ ). The supersymmetric level  $k$   $SU(2)$  WZW theory includes these three fermions plus a bosonic WZW model of level  $(k-2)$ . The operators  $\Phi_{j;m,\bar{m}}^{(su)}$  (with  $2j = 0, 1, \dots, k-2$ ;  $m, \bar{m} = -j, -j+1, \dots, j$ ) are primaries of the bosonic  $SU(2)_{k-2}$  WZW model, whose dimension is

$$\Delta_j^{(su)} = \bar{\Delta}_j^{(su)} = \frac{j(j+1)}{k}. \quad (4.12)$$

The notation  $(\psi\bar{\psi}\Phi_j^{(su)})_{j+1;m,\bar{m}}$  means that we are coupling the fermions and bosons into a primary of spin  $j+1$  and  $(J_3^{(\text{tot})}, \bar{J}_3^{(\text{tot})}) = (m, \bar{m})$  in the supersymmetric  $SU(2)_k$  WZW model. For example, for the special case  $m = j+1$  that will be useful below, one has (in a natural overall normalization):

$$\begin{aligned} (\psi\bar{\psi}\Phi_j^{(su)})_{j+1;j+1,\bar{m}} &= \frac{1}{\sqrt{(2j+1)(2j+2)}}\psi^+[\sqrt{(j+\bar{m})(j+\bar{m}+1)}\bar{\psi}^+\Phi_{j;j,\bar{m}-1}^{(su)} \\ &\quad + \sqrt{2(j+\bar{m}+1)(j-\bar{m}+1)}\bar{\psi}^3\Phi_{j;j,\bar{m}}^{(su)} + \sqrt{(j-\bar{m})(j-\bar{m}+1)}\bar{\psi}^-\Phi_{j;j,\bar{m}+1}^{(su)}]. \end{aligned} \quad (4.13)$$



The physical state condition (3.8) requires that the vertex operator (4.11) satisfy

$$Q^2(\tilde{j} - j)(\tilde{j} + j + 1) = p^2 . \quad (4.14)$$

On the right hand side of (4.11),  $X^i$  with  $i = 6, 7, 8, 9$  are the four scalar fields in the adjoint of  $SU(k)$  which parametrize the locations of the fivebranes in the transverse directions. To match to the representation of  $SO(4) \simeq SU(2)_L \times SU(2)_R$  that appears on the left hand side, one must restrict to symmetric traceless tensors in  $(i_1, i_2, \dots, i_{2j+2})$ .

The notation  $\widetilde{\text{tr}}$  refers to the fact that the gauge theory operator which appears on the right-hand side of (4.11) is a combination of single and multi-trace operators. We will normalize  $\widetilde{\text{tr}}$  such that the single-trace component has a coefficient equal to one. The relative normalization between the left and right hand sides of (4.11) will be discussed in section 5.

The fact that single string vertex operators correspond to combinations of single and multi-trace operators is expected to be generic in holographic dualities, but the precise combinations are in general unknown. The identification (4.11) is based on the fact that the two sides transform in the same chiral representation of the supersymmetry algebra, but this does not enable us to distinguish the single-trace from the multi-trace operators (which transform in the same way under supersymmetry). Interestingly, in the case of  $NS5$ -branes, we will be able to determine the precise combination of multi-trace operators in the gauge theory that corresponds to the string theory vertex operators (in §6).

We will now further restrict the discussion to the operators

$$\mathcal{O}_{j+1-\bar{m}, j+1+\bar{m}}(p_\mu) \equiv e^{-\varphi-\bar{\varphi}}(\psi\bar{\psi}\Phi_j^{(su)})_{j+1; j+1, \bar{m}} e^{Q\tilde{j}\phi} e^{ip_\mu x^\mu} \leftrightarrow \widetilde{\text{tr}}(A^{j+1-\bar{m}} B^{j+1+\bar{m}})(p_\mu) . \quad (4.15)$$

The identification with the gauge theory operators in (4.15) relies on an identification of the rotation groups  $SO(2)_A, SO(2)_B$  with particular subgroups of  $SU(2)_L \times SU(2)_R$  in the geometry (4.1). This identification and other aspects of the operator maps are discussed in more detail in [3].

So far we discussed the form of the vertex operators in the unresolved CHS geometry (4.1), or equivalently in the full resolved geometry (4.7) but far from the tip of the cigar. In order to compute correlation functions, we will need the form of the full vertex operators in the coset theory. It is convenient to discuss separately the cases  $|\bar{m}| = j+1$  and  $|\bar{m}| \leq j$ .

For  $\bar{m} = j+1$  we expect (4.15) that  $\mathcal{O}_{0, 2j+2} \sim \widetilde{\text{tr}}(B^{2j+2})$ , and we have

$$\mathcal{O}_{0, 2j+2}(p_\mu) = e^{-\varphi-\bar{\varphi}} \psi^+ \bar{\psi}^+ \Phi_{j; j, j}^{(su)} e^{Q\tilde{j}\phi} e^{ip_\mu x^\mu} . \quad (4.16)$$

In order to write  $\mathcal{O}_{0,2j+2}$  as a vertex operator in the full geometry (4.7), we would like to decompose the  $SU(2)$  WZW part of the vertex operator (4.16) into its  $U(1)$  and  $\frac{SU(2)}{U(1)}$  components. It is not difficult to show that

$$\psi^+ \bar{\psi}^+ \Phi_{j;j,j}^{(su)} = e^{iQ(j+1)(Y-\bar{Y})} V_{\frac{k}{2}-j-1;-\frac{k}{2}+j+1,-\frac{k}{2}+j+1}^{(su,susy)} \quad (4.17)$$

Here,  $Y, \bar{Y}$  are the left and right moving parts of the worldsheet field corresponding to the compact coordinate around the cigar. In particular, we see that for generic  $j$ , the vertex operator (4.17) carries fractional winding around the circle labeled by  $Y$ . As mentioned earlier, this is possible due to the  $\mathbf{Z}_k$  orbifold in (4.7).

The operators  $V_{j;m,\bar{m}}^{(su,susy)}$  can be defined (as in the  $SL(2)$  discussion above) by starting with the (supersymmetric)  $SU(2)$  vertex operator  $\Phi_{j;m,\bar{m}}^{(su)}$  and removing from it the  $U(1)$  part. In particular,  $V_{j;m,\bar{m}}^{(su,susy)}$  has dimension and  $R$ -charge

$$\begin{aligned} \Delta_{j;m}^{(su)} &= \frac{j(j+1) - m^2}{k}; & \bar{\Delta}_{j;\bar{m}}^{(su)} &= \frac{j(j+1) - \bar{m}^2}{k}, \\ R_m^{(su)} &= -\frac{2m}{k}; & \bar{R}_{\bar{m}}^{(su)} &= -\frac{2\bar{m}}{k}. \end{aligned} \quad (4.18)$$

Plugging (4.17) in (4.16), and using (3.6), we find that

$$\mathcal{O}_{0,2j+2}(p_\mu) = e^{-\varphi - \bar{\varphi}} V_{\frac{k}{2}-j-1;-\frac{k}{2}+j+1,-\frac{k}{2}+j+1}^{(su,susy)} V_{\tilde{j};j+1,j+1}^{(sl,susy)} e^{ip_\mu x^\mu}. \quad (4.19)$$

For the case  $\bar{m} = -(j+1)$  a similar discussion leads to the result

$$\mathcal{O}_{2j+2,0}(p_\mu) = e^{-\varphi - \bar{\varphi}} V_{\frac{k}{2}-j-1;-\frac{k}{2}+j+1,\frac{k}{2}-j-1}^{(su,susy)} V_{\tilde{j};j+1,-(j+1)}^{(sl,susy)} e^{ip_\mu x^\mu}. \quad (4.20)$$

In the gauge theory, the operator (4.20) should correspond to  $\widetilde{\text{tr}}(A^{2j+2})$  (4.15).

Next we turn to vertex operators (4.15) with  $-j \leq \bar{m} \leq j$ . In general, there are now three terms in the expansion (4.13). The terms proportional to  $\bar{\psi}^\pm$  in (4.13) can be described as follows. Consider the operator

$$\psi^+ \bar{\psi}^\pm \Phi_{j;j,\bar{m} \mp 1}^{(su)} \Phi_{\tilde{j};j+1,\bar{m}}^{(sl)} \quad (4.21)$$

in SCFT on  $SU(2) \times SL(2, \mathbb{R})$ . This operator has the property that if we decompose it under  $[\frac{SU(2)}{U(1)} \times \frac{SL(2, \mathbb{R})}{U(1)}] \times [U(1)^2]$ , the  $\frac{SU(2)}{U(1)} \times \frac{SL(2, \mathbb{R})}{U(1)}$  component is precisely the operator we are interested in, whose asymptotic form in the CHS geometry is

$$\psi^+ \bar{\psi}^\pm \Phi_{j;j,\bar{m} \mp 1}^{(su)} e^{Q\tilde{j}\phi}, \quad (4.22)$$

while the  $U(1)^2$  component has dimension zero and has the form  $e^{iQmZ}$  with  $Z$  a null scalar field (a combination of the scalar fields associated with the two  $U(1)$ 's). The only effect of the  $U(1)^2$  component is to impose the conservation of  $U(1)^2$  charges, which we will require in our calculations anyway. Thus, the  $U(1)^2$  part of the vertex operator (4.21) plays no role in the calculations.

As we saw in §2, the analytic structure of amplitudes is largely determined by the  $SL(2)/U(1)$  part of the vertex operators. We will see later (in §5.3) that the  $SL(2)/U(1)$  part of (4.21) does not lead to singularities at low energies. Thus, the contributions (4.21), (4.22) to the vertex operator (4.15) can in fact be neglected at low energies.

The remaining contribution to the vertex operator (4.15) is proportional to

$$\psi^+ \bar{\psi}^3 \Phi_{j;\tilde{j},\tilde{m}}^{(su)} e^{Q\tilde{j}\phi}. \quad (4.23)$$

In order to write it in a form similar to that given above for the other terms, we would like to decompose (4.23) into its  $SL(2, \mathbb{R})/U(1)$  and  $SU(2)/U(1)$  components, and then lift the results to  $SL(2, \mathbb{R}) \times SU(2)$ . The first thing to note is that  $\bar{\psi}^3$  belongs to the  $SL(2, \mathbb{R})$  component. It is in fact identical to  $\bar{\psi}_Y$  discussed after equation (3.6). In the notation of §3, we can write the right-moving,  $SL(2, \mathbb{R})/U(1)$  part of the vertex operator (4.23) as

$$\bar{\psi}^3 e^{Q(\tilde{j}\phi - i\tilde{m}\bar{Y})} = \bar{\psi}_Y e^{Q(\tilde{j}\phi - i\tilde{m}\bar{Y})}. \quad (4.24)$$

As explained in §3, this operator is a descendant of the  $\mathcal{N} = 2$  primary  $e^{Q(\tilde{j}\phi - i\tilde{m}\bar{Y})}$ . Using the form of the  $\mathcal{N} = 2$  superconformal generators far from the tip of the cigar,

$$\begin{aligned} \bar{G}^+ &= (\bar{\psi}_\phi + i\bar{\psi}_Y) \partial(\phi + i\bar{Y}) + Q \partial(\bar{\psi}_\phi + i\bar{\psi}_Y), \\ \bar{G}^- &= (\bar{\psi}_\phi - i\bar{\psi}_Y) \partial(\phi - i\bar{Y}) + Q \partial(\bar{\psi}_\phi - i\bar{\psi}_Y), \end{aligned} \quad (4.25)$$

one can show that

$$\bar{\psi}_Y e^{Q(\tilde{j}\phi + i(mY - \tilde{m}\bar{Y}))} \sim \left( \frac{1}{\tilde{j} - \tilde{m}} \bar{G}_{-\frac{1}{2}}^+ - \frac{1}{\tilde{j} + \tilde{m}} \bar{G}_{-\frac{1}{2}}^- \right) V_{\tilde{j};m,\tilde{m}}^{(sl,susy)}. \quad (4.26)$$

A natural question at this point is how can the operator on the left hand side of (4.26), which seems to be regular as  $\tilde{j} \rightarrow \pm\tilde{m}$ , be the same as the operator on the right hand side, which diverges in this limit. The answer is that on the left hand side we only wrote the leading, non-normalizable, contribution to the operator which is defined algebraically on the right hand side. That contribution is indeed finite in the limit  $\tilde{j} \rightarrow \pm\tilde{m}$  also on the right-hand side (one can show that  $\bar{G}_{-\frac{1}{2}}^+$  annihilates the non-normalizable component of  $V_{\tilde{m};m,\tilde{m}}$

and that  $\bar{G}_{-\frac{1}{2}}^-$  annihilates the non-normalizable component of  $V_{-\bar{m};m,\bar{m}}$ . The divergence of the operator is reflected in the divergence of the leading normalizable contribution, *i.e.* the poles as  $\tilde{j} \rightarrow \pm\bar{m}$  are potential LSZ poles. We will see later (in §5.3) that a more careful analysis of these poles leads to a picture which is in agreement with expectations.

One can also write an operator in SCFT on  $SL(2, \mathbb{R}) \times SU(2)$  which has the property that removing from it the  $U(1)^2$  components leads to the operator (4.23):

$$\psi^+ \left( \frac{1}{\tilde{j} - \bar{m}} \bar{G}_{-\frac{1}{2}}^+ - \frac{1}{\tilde{j} + \bar{m}} \bar{G}_{-\frac{1}{2}}^- \right) \Phi_{j;\tilde{j},\bar{m}}^{(su)} \Phi_{\tilde{j};j+1,\bar{m}}^{(sl)}. \quad (4.27)$$

Here  $\bar{G}_{-\frac{1}{2}}^\pm$  are operators defined in the full  $SL(2, \mathbb{R})$  CFT, but they commute with the  $U(1)$ . Their description in terms of  $SL(2)$  currents is reviewed in appendix A.

This concludes our brief review of six dimensional DSLST. In the next subsection we will describe the LSZ poles in this theory and compare the resulting pattern with the low-energy gauge theory. In the following two sections, we will use our results to analyze some correlation functions in this theory, and in particular their analytic structure at small momenta.

#### 4.2. LSZ poles

In this subsection we analyze the LSZ poles associated with the operators (4.15), and compare the low-energy pole structure with our expectations from the gauge theory. In the gauge theory, operators are expected to exhibit poles only if they can create single-particle states. The operators (4.11), (4.15) obviously couple to a state with  $2j + 2$  massless particles. However, if  $2j + 1$  of the  $2j + 2$  fields appearing in the operator are  $B$ 's (or  $B^*$ 's), then at the point (4.4) in the moduli space we can replace these  $2j + 1$  fields by their vacuum expectation value (VEV), and obtain an operator that can create from the vacuum a single-particle state involving the remaining field. Thus, we expect to find LSZ poles for the operators (4.15) with  $\bar{m} = j, j+1$ , corresponding to the gauge theory operators  $\tilde{\text{tr}}(AB^{2j+1})$ ,  $\tilde{\text{tr}}(B^{2j+2})$ , respectively, but not for the other values of  $\bar{m}$ .

Let us start with the case  $\bar{m} = j+1$ , corresponding to the vertex operator (4.19). The analysis of §2 and §3 shows that this operator has LSZ poles at  $\tilde{j} = j, j-1, j-2, \dots > -1/2$ . The mass-shell condition (4.14) maps these to momentum-space poles occurring at (for  $\alpha' = 2$ )

$$p^2 = 0, \quad p^2 = -\frac{4j}{k}, \quad p^2 = -\frac{4(2j-1)}{k}, \quad \dots, \quad (4.28)$$

respectively. Thus, we find that this operator has a massless LSZ pole for all  $j \geq 0$ , in agreement with the field theory expectations described above.

For the case  $\bar{m} = -(j+1)$ , it is clear from (4.20) and from the discussion of §2, §3 that the operator  $\mathcal{O}_{2j+2,0}$  has no LSZ poles. Again, the absence of a massless LSZ pole is consistent with the fact that the corresponding gauge theory operator,  $\widetilde{\text{tr}}(A^{2j+2})$ , does not couple to single particle massless states. Interestingly, we find that it does not couple to massive single particle states localized near the tip of the cigar either.

It remains to discuss the case  $|\bar{m}| \leq j$ , for which the operators in question are obtained from (4.27) by removing the  $U(1)^2$  part. For  $|\bar{m}| < j$  the situation is simple. Massless poles would again have to appear at  $\tilde{j} = j$  (4.14), but this case does not belong to the set (2.22), (2.31), so these operators do not create massless states when acting on the vacuum, although some of them do create massive states. Again, this is compatible with the gauge theory expectations, since all these operators contain at least two  $A$ 's and thus do not couple to single particle massless states.

For  $|\bar{m}| = j$  the situation is slightly more subtle, because of the explicit factors of  $\tilde{j} \pm \bar{m}$  in the denominator of (4.27), which as we explained above, can potentially lead to (massless) LSZ poles. Repeating the analysis of §2 we find<sup>17</sup> that while the pole at  $\tilde{j} = \bar{m} = j$  does create a massless particle from the vacuum, and thus is an LSZ pole, the pole at  $\tilde{j} = -\bar{m} = j$  does not correspond to an LSZ pole, and is analogous to the poles associated with  $\Gamma(-2j-1)$  in the reflection coefficient (2.7). This is again consistent with the gauge theory expectations, since the operators (4.15) with  $\bar{m} = j$  correspond to  $\widetilde{\text{tr}}(AB^{2j+1})$  and thus should couple to single particle states, while those with  $\bar{m} = -j$  correspond to  $\widetilde{\text{tr}}(A^{2j+1}B)$ , and should not.

Thus, for all the operators discussed here we find a precise agreement between the LSZ poles found in string theory and our low-energy field theory expectations. We have verified that this is true also for some additional operators (including Ramond-Ramond (RR) sector operators [3]).

---

<sup>17</sup> There are two ways to repeat this analysis here. One possibility is to decompose the operators of the SCFT on  $SL(2)/U(1)$  in terms of operators in a *bosonic*  $SL(2)$  theory, and then the analysis of §2 may be applied directly and it is straightforward to see that a massless pole originating by taking  $x \rightarrow 0, \infty$  in an  $n$  point function appears for an operator in (4.15) iff it should appear in the low energy QFT via the LSZ reduction. Alternatively, we can continue working using the supersymmetric LST, by writing the operators  $\bar{G}^\pm$  in terms of  $SL(2)$  currents  $\bar{J}^\pm$  which act on the  $x$ -variables in a known way (as an  $\bar{x}$  derivative, see [21] and references therein), and analyzing whether the regions  $x \rightarrow 0, \infty$  give poles or not. This leads, of course, to the same answers.

## 5. Some examples of correlation functions

In this section we will study some simple examples of correlation functions of operators in six dimensional DSLST, focusing on their low energy behavior. Some puzzles regarding these correlation functions were encountered in [14]. We will use the discussion of the previous sections to clarify their analytic structure, and in particular to resolve the aforementioned puzzles.

We will see that, even at low momenta, some of the poles exhibited by these amplitudes are due to dynamics that is not captured by the  $U(1)^{k-1}$  IR free gauge theory that one expects to describe  $k$  separated type IIB NS fivebranes at long distances. In the first three subsections we discuss examples of two-point functions, and show that they generally receive contributions both from bulk poles and from the LSZ poles discussed in §4.2. In §5.4 we discuss the double pole terms in a specific non-trivial three-point function. In §5.5 we review an S-matrix computation performed in [3], and in §5.6 we comment on some winding-number-violating correlation functions.

### 5.1. $\langle \widetilde{\text{tr}}(B^n) \widetilde{\text{tr}}((B^*)^n) \rangle$

The string theory vertex operator with the quantum numbers of  $\text{tr}(B^n)$  is  $\mathcal{O}_{0,n}$  (4.16), (4.19). As discussed earlier,  $\mathcal{O}_{0,n}$  actually corresponds (in the low-energy gauge theory) to a mixture of single and multi-trace operators,

$$\mathcal{O}_{0,n} = C_{n,k} \frac{1}{n} \widetilde{\text{tr}}(B^n), \quad (5.1)$$

where  $C_{n,k}$  is a normalization constant to be determined, and the operator  $\widetilde{\text{tr}}(B^n)$  is a specific linear combination of  $\text{tr}(B^n)$  and multi-trace operators, which will be determined in the next section.

In the  $U(1)^{k-1}$  gauge theory at the point (4.4) in its moduli space, the leading (in  $1/M_W$ ) contribution to the two-point function of  $\widetilde{\text{tr}}(B^n)$  comes from the single trace term:

$$\langle \frac{1}{n} \widetilde{\text{tr}}(B^n) \frac{1}{n} \widetilde{\text{tr}}((B^*)^n) \rangle = \langle \frac{1}{n} \text{tr}(B^n) \frac{1}{n} \text{tr}((B^*)^n) \rangle \simeq \frac{k M_W^{2n-2}}{p_\mu^2}, \quad (5.2)$$

where we have contracted a single  $B$  with a single  $B^*$ , and replaced the remaining  $B$ 's and  $B^*$ 's by their VEV (4.4) (setting  $M_s = 1$  in the process). The multi-trace terms in  $\widetilde{\text{tr}}(B^n)$  do not contribute at this order in the  $1/M_W$  expansion, since they involve factors of  $\langle \text{tr}(B^l) \rangle$  with  $0 < l < n$ , which vanish.

The string theory analog of (5.2) involves a two-point function of the operators  $\mathcal{O}_{0,2j+2}$ , (4.16), (4.19). For small  $p_\mu^2$ , the mass shell condition (4.14) implies that

$$\tilde{j} - j \simeq \frac{k}{2} \frac{p_\mu^2}{2j+1}. \quad (5.3)$$

The two-point function of  $\mathcal{O}_{0,2j+2}$  is thus given by the product of the  $SU(2)/U(1)$  and  $SL(2, \mathbb{R})/U(1)$  contributions (the  $\mathbb{R}^{5,1}$  contribution is equal to one). We will normalize the  $SU(2)/U(1)$  vertex operators  $V^{(su, susy)}$  in (4.19) such that their two-point function is also equal to one. Thus, we have<sup>18</sup>

$$\langle \mathcal{O}_{0,2j+2}(p_\mu) \mathcal{O}_{0,-(2j+2)}(-p_\mu) \rangle = \frac{1}{g_s^2} \langle V_{\tilde{j}; j+1, j+1}^{(sl, susy)} V_{\tilde{j}; -(j+1), -(j+1)}^{(sl, susy)} \rangle. \quad (5.4)$$

We have inserted a factor of  $\frac{1}{g_s^2}$  to account for the fact that we are computing an amplitude on the sphere.  $g_s$  is proportional to the string coupling at the tip of the cigar, (4.8). The general formula for the two-point function in  $SL(2, \mathbb{R})/U(1)$  appears in appendix A. It indeed has a pole at  $\tilde{j} = j$  (or  $p_\mu^2 = 0$ , (5.3)), near which it behaves as

$$\langle \mathcal{O}_{0,2j+2}(p_\mu) \mathcal{O}_{0,-(2j+2)}(-p_\mu) \rangle \simeq \frac{1}{g_s^2} \left( \frac{2j+1}{k} \right)^2 \frac{1}{p_\mu^2}. \quad (5.5)$$

Combining (5.1), (5.2) and (5.5) we seem to conclude that

$$C_{2j+2,k}^2 k M_W^{4j+2} = \frac{1}{g_s^2} \left( \frac{2j+1}{k} \right)^2. \quad (5.6)$$

Comparing to (4.8) we see that the solution of this equation is

$$\begin{aligned} \frac{1}{g_s^2} &= C(k) M_W^2, \\ C_{2j+2,k} &= \frac{2j+1}{k M_W^{2j}} \sqrt{\frac{C(k)}{k}} \end{aligned} \quad (5.7)$$

for some function  $C(k)$ . This function can be determined by a careful computation of three-point functions, but we will not need its explicit form here.

Actually, the discussion above (and a similar discussion which appeared in §5.1 of [3]) contains a subtle flaw, which turns out to be relatively benign in this case, but plays an important role in understanding other correlation functions. The issue is whether it is

---

<sup>18</sup> The operator corresponding to the complex conjugate of  $\tilde{\text{tr}} B^{2j+2}$  is  $\mathcal{O}_{0,-(2j+2)} \sim \tilde{\text{tr}}(B^*)^{2j+2}$ .

really the case that the behavior of the correlation function (5.5) near the pole at  $p_\mu^2 = 0$  is entirely due to the low energy gauge theory contribution described above, or whether it receives additional contributions.

To answer this question using the techniques of the previous sections we need to take a closer look at the two-point function (5.4). The analytic structure as a function of  $\tilde{j}$  is due to the behavior of the integral (see (2.19))

$$\int d^2x |x|^{2(\tilde{j}+j+1)} |1-x|^{-4(\tilde{j}+1)} = \pi \gamma(-2\tilde{j}-1) \gamma(\tilde{j}-j) \gamma(\tilde{j}+j+2), \quad (5.8)$$

where  $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$ . Taking  $\tilde{j} = j + \epsilon$  and studying the right-hand side of (5.8) in the limit  $\epsilon \rightarrow 0$ , one finds that the integral behaves like  $\pi/2\epsilon$ ; this was used to arrive at (5.5). As we saw in the previous sections, the contribution to the residue of the pole at  $\epsilon \rightarrow 0$  due to a normalizable state created from the vacuum by the operator (4.19) comes from the region near  $x = 0$  or  $x = \infty$  in the integral (5.8). In our case, the relevant region is  $x \rightarrow \infty$ , where one has (as  $\epsilon \rightarrow 0$ )

$$\int^\infty d^2x |x|^{2(j-\tilde{j}-1)} \simeq \frac{\pi}{\epsilon}. \quad (5.9)$$

Thus, we see that the contribution of the normalizable discrete state leads to a pole with a residue that is twice as large as the one computed for the full integral.

What cancels half of the contribution of this normalizable state? On the level of the integral (5.8), the answer is clear. As  $\tilde{j} \rightarrow j$ , in addition to the contribution from  $x \rightarrow \infty$ , there is a further divergent contribution coming from  $x \rightarrow 1$ , which also gives a pole (since  $2j$  is a non-negative integer). This contribution can be computed by methods similar to the ones we used in §2, and gives  $-\pi/2\epsilon$ . Together, the two contributions account for the behavior of the full integral.

The physical interpretation of the two terms is clear as well, following the discussion of the previous sections. The contribution to the residue of the pole from  $x \rightarrow \infty$  should be interpreted as due to the overlap of the string theory vertex operator (4.19) with the state created by the operator  $\widetilde{\text{tr}}(B^{2j+2})$  in the low energy gauge theory. This contribution is always positive, as required by unitarity. The contribution from  $x \rightarrow 1$  (corresponding to  $x_1 \rightarrow x_2$  before we rescale the  $x$ 's, as in (2.59)) has a non-gauge theoretic origin. In the deformed CHS geometry (4.7) it is associated with processes that occur in the bulk of the cigar, very far from the tip. Indeed, since the pole occurs when  $2j$  is an integer, the



two-point function in question satisfies the condition for bulk poles (2.49) (with  $l = n = 2$  and  $g = n_2 = 0$ ).

We see that the correlation function (5.4) receives two kinds of large contributions at low energies. One can be interpreted as due to the dynamics of the low energy  $U(1)^{k-1}$  gauge theory, and comes from the region near the tip of the cigar; the other is due to dynamics in the bulk of the cigar, and has a non-gauge theoretic origin. It is interesting that the bulk of the cigar can give a massless pole in the correlation function, even though all the physical states that live there have masses obeying  $m^2 \geq 1/k\alpha'$ . Obviously, such an effect could not happen in a standard field theory, where a pole in the two-point function would necessarily be associated with creating a particle of the appropriate mass. The non-local nature of LST seems to play an important role in making this possible here.

The above discussion implies that our analysis of the relation between the vertex operators  $\mathcal{O}_{0,2j+2}$  and the gauge theory operators  $\widetilde{\text{tr}}(B^{2j+2})$  must be modified slightly. In particular, the right-hand side of equation (5.5) must be multiplied by a factor of two, since it should only contain the contribution to the residue of the pole from  $x \rightarrow \infty$ . This factor of two will not play an important role below, since we will not attempt to keep track of numerical coefficients (but only of the  $j$  dependence). It is nevertheless interesting that the behavior of this amplitude at low momenta is not fully accounted for by the  $U(1)^{k-1}$  gauge theory that lives on the separated fivebranes. We next turn to a more dramatic manifestation of this phenomenon.

## 5.2. $\langle \widetilde{\text{tr}}(A^n) \widetilde{\text{tr}}((A^*)^n) \rangle$

In this subsection we will repeat the discussion of the previous subsection for the operator  $\mathcal{O}_{n,0}$  (4.20), which has the quantum numbers of  $\text{tr}(A^n)$  in the low energy gauge theory. Again, the vertex operator  $\mathcal{O}_{n,0}$  actually corresponds to a mixture of single and multi-trace operators in the gauge theory, but like in the previous subsection, this will not play an important role in our discussion.

Since the scalar field  $A$  does not have an expectation value at the point in moduli space where we are working, (4.4), the gauge theory calculation is even simpler than (5.2) in this case. At order  $M_W^{2n-2}$ , the two-point function  $\langle \widetilde{\text{tr}}(A^n) \widetilde{\text{tr}}((A^*)^n) \rangle$  is exactly zero. So are all other contributions to this two-point function that go like  $M_W^{2l}$ , with  $l > 0$ .

It is thus interesting to compute the two-point function of  $\mathcal{O}_{n,0}$  in string theory. The fact that in the  $U(1)^{k-1}$  gauge theory, the contributions that go like  $M_W^{2(n-1-l)}$  with  $l = 0, 1, 2, \dots, n-2$  vanish, would lead one to expect that in string theory in the background

(4.7) the genus  $l < n - 1$  contributions to the two-point function of  $\mathcal{O}_{n,0}$  (at low energies) vanish as well. Any non-vanishing contributions at low genus would have to have a non-gauge theoretic origin.

The contribution of  $\mathbb{R}^{5,1}$  and  $SU(2)/U(1)$  to the two-point function of the operators (4.20) is again equal to one in the conventional choice of normalizations on the compact coset. The two-point function in  $SL(2, \mathbb{R})/U(1)$  can be read off the general formula in appendix A,

$$\langle V_{\tilde{j};j+1,-(j+1)}^{(sl,susy)} V_{\tilde{j};-(j+1),j+1}^{(sl,susy)} \rangle = \frac{2\tilde{j}+1}{k} \gamma(-2\tilde{j}-1) \frac{\Gamma^2(\tilde{j}-j)}{\Gamma^2(-\tilde{j}-j-1)}. \quad (5.10)$$

Using the physical state condition (5.3) we find that not only is the tree level contribution to the two-point function (5.10) non-zero, it in fact has a pole at  $p_\mu^2 = 0$  [14],

$$\langle \mathcal{O}_{2j+2,0}(p_\mu) \mathcal{O}_{-(2j+2),0}(-p_\mu) \rangle \simeq \frac{1}{g_s^2} (-1)^{2j} \left( \frac{2j+1}{k} \right)^2 \frac{1}{p_\mu^2}. \quad (5.11)$$

The alternating sign of the residue makes it clear that this pole cannot in general be interpreted as due to an on-shell one particle state, which is just as well, since we know that the operator  $\text{tr}(A^n)$  should not create such states. Indeed, the analysis of §2 shows that vertex operators such as (4.20) do not have LSZ poles in their correlation functions.

To understand the physical interpretation of the pole (5.11) we go back to the representation of the two-point function (5.10) in terms of an integral over  $x$ ,

$$\int d^2x x^{\tilde{j}+j+1} \bar{x}^{\tilde{j}-j-1} |1-x|^{-4(\tilde{j}+1)} = \pi \gamma(-2\tilde{j}-1) \frac{\Gamma^2(\tilde{j}-j)}{\Gamma^2(-\tilde{j}-j-1)}. \quad (5.12)$$

This integral exhibits a pole as  $\tilde{j} \rightarrow j$ , and we would like to understand its origin. It is easy to see that the integral is well behaved as  $x \rightarrow 0, \infty$ . The divergence is in this case entirely due to the behavior as  $x \rightarrow 1$ . As explained in the previous sections, such divergences are due to bulk processes.

We conclude that for the operators  $\mathcal{O}_{n,0}$ , which have the quantum numbers of  $\tilde{\text{tr}}(A^n)$ , the full string theory tree-level two-point function, which has a pole at vanishing  $p_\mu^2$ , is non-zero due to effects that cannot be seen in the low energy  $U(1)^{k-1}$  gauge theory of  $k$  separated fivebranes.

### 5.3. $\langle \widetilde{\text{tr}}(A^n B^{2j+2-n}) \widetilde{\text{tr}}((A^*)^n (B^*)^{2j+2-n}) \rangle$

The vertex operators corresponding to gauge theory operators that include both  $A$  and  $B$  are given by (4.15), (4.13), (4.27). For  $|\bar{m}| < j$  it is easy to check that the two-point function is finite as  $p^2 \rightarrow 0$  (or  $\tilde{j} \rightarrow j$ ). This leaves the cases  $\bar{m} = \pm j$  corresponding to the gauge theory operators  $\mathcal{O}_{1,2j+1} \sim \widetilde{\text{tr}}(AB^{2j+1})$  and  $\mathcal{O}_{2j+1,1} \sim \widetilde{\text{tr}}(A^{2j+1}B)$ .

Consider the case  $\bar{m} = j$ . As we saw in §4, the vertex operator  $\mathcal{O}_{1,2j+1}$  has two terms. One, whose asymptotic form far from the tip of the cigar is given by (4.22), has the property that its  $SL(2)/U(1)$  part is proportional to  $V_{\tilde{j},j+1;j}^{(sl,susy)}$ . Using the results of appendix A for the two-point function, it is not difficult to see that this term has a two-point function that is regular as  $p^2 \rightarrow 0$ . The other term, whose asymptotic form looks like (4.23), can be obtained from the  $SU(2) \times SL(2)$  operator (4.27) by removing the  $U(1)^2$  part, as explained in §4.

The two-point function of this operator is proportional to

$$\frac{1}{(\tilde{j} - j)(\tilde{j} + j)} \langle V_{\tilde{j},-j-1;-j}^{(sl,susy)} \{ \bar{G}_{-\frac{1}{2}}^-, \bar{G}_{-\frac{1}{2}}^+ \} V_{\tilde{j},j+1;j}^{(sl,susy)} \rangle. \quad (5.13)$$

Using the fact that the anti-commutator  $\{ \bar{G}_{-\frac{1}{2}}^-, \bar{G}_{-\frac{1}{2}}^+ \}$  is proportional to  $\bar{L}_{-1}$ , which acts as a worldsheet derivative on  $\langle V_{\tilde{j},-j-1;-j}^{(sl,susy)} V_{\tilde{j},j+1;j}^{(sl,susy)} \rangle$ , we find that (5.13) is proportional to

$$\frac{\Delta_{\tilde{j};j}}{(\tilde{j} - j)(\tilde{j} + j)} \langle V_{\tilde{j},-j-1;-j}^{(sl,susy)} V_{\tilde{j},j+1;j}^{(sl,susy)} \rangle, \quad (5.14)$$

where  $\Delta_{\tilde{j};j}$  is the scaling dimension (3.3). In the limit  $\tilde{j} \rightarrow j$ , which corresponds via (5.3) to  $p^2 \rightarrow 0$ , we find a single pole, whose residue is readily computable.

In the case  $\bar{m} = -j$  one again finds a pole in the two point function, however, as we saw in §4.2, it should be thought of as a bulk pole, and not an LSZ pole.

### 5.4. $\langle \widetilde{\text{tr}}(F_{\mu\nu} B^{n_1}) \widetilde{\text{tr}}(F_{\mu\nu} B^{n_2}) \widetilde{\text{tr}}((B^*)^{n_1+n_2}) \rangle$

As another test of our techniques we next turn to a correlation function involving the Ramond-Ramond vertex operators  $\mathcal{O}_n^+$  discussed in [3]. These operators have the quantum numbers of

$$\mathcal{O}_n^+ \sim \xi^{\mu\nu} \widetilde{\text{tr}}(F_{\mu\nu} B^n), \quad (5.15)$$

where, again, the left-hand side is a vertex operator in the background (4.7), while the right-hand side is a combination of single and multi-trace operators in the low-energy gauge

theory. As in the previous subsections, the multi-trace components will not play a role in our calculations.

Consider first the leading contribution to the connected three-point function in the low energy gauge theory:

$$\begin{aligned} \xi_{\mu\nu}^{(1)} \xi_{\mu'\nu'}^{(2)} \langle \text{tr}(F^{\mu\nu} B^{n_1})(p_1) \text{tr}(F^{\mu'\nu'} B^{n_2})(p_2) \frac{1}{n_1 + n_2} \text{tr}((B^*)^{n_1+n_2})(p_3) \rangle = \\ \delta^6(p_1 + p_2 + p_3) k M_W^{2(n_1+n_2-1)} \frac{\xi_{\mu\nu}^{(1)} \xi_{\mu'\nu'}^{(2)}}{p_3^2} \left[ n_2 \frac{p_1^\mu p_1^{\mu'} \eta^{\nu\nu'} \pm (\mu \leftrightarrow \nu, \mu' \leftrightarrow \nu')}{p_1^2} + (1 \leftrightarrow 2) \right]. \end{aligned} \quad (5.16)$$

It comes from a contraction of the two  $F$ 's, and a second contraction of one of the  $B$ 's with one of the  $(B^*)$ 's.

We would like to compute the same object in string theory, and in particular reproduce the pole structure and the dependence on  $n_1, n_2$  in (5.16). The vertex operators  $\mathcal{O}_n^+$  in the  $(-1/2, -1/2)$  picture are given in equation (5.4) in [3]. Substituting them into the three point function

$$\langle \mathcal{O}_{2j_1+1}^+(p_1) \mathcal{O}_{2j_2+1}^+(p_2) \mathcal{O}_{0,-2(j_1+j_2)-2}(p_3) \rangle \quad (5.17)$$

one finds two terms. One goes like  $p_1^\rho p_1^\sigma$ ; the other, like  $p_2^\rho p_2^\sigma$ . This is consistent with the gauge theory answer (5.16), which has two terms related by interchanging  $1 \leftrightarrow 2$ . Thus, we focus on the term that goes like  $p_1^\rho p_1^\sigma$ .

In this term one has [3] asymptotically

$$\begin{aligned} \mathcal{O}_{2j_1+1}^+ &\simeq \frac{\xi_{\mu\nu}^{(1)} \gamma_{a\bar{a}}^{\mu\nu} e^{-\frac{1}{2}(\varphi+\bar{\varphi})+\frac{i}{2}(H+\bar{H})}}{Q^2(j_1+\tilde{j}_1+1)^2} (\gamma_\rho)_a^{\dot{c}} p_1^\rho S_{\dot{c}}^c(\gamma_\sigma)_a^{\dot{c}} p_1^\sigma \bar{S}_c^{\dot{c}} e^{-\frac{i}{2}(H'+\bar{H}')} \Phi_{j_1;j_1,j_1}^{(su)} e^{Q\tilde{j}_1\phi+ip_1\cdot x}, \\ \mathcal{O}_{2j_2+1}^+ &\simeq \xi_{\mu'\nu'}^{(2)} \gamma_{b\bar{b}}^{\mu'\nu'} e^{-\frac{1}{2}(\varphi+\bar{\varphi})+\frac{i}{2}(H+\bar{H}+H'+\bar{H}')} S_b^{\dot{c}} \bar{S}_{\dot{c}}^b \Phi_{j_2;j_2,j_2}^{(su)} e^{Q\tilde{j}_2\phi+ip_2\cdot x}, \\ \mathcal{O}_{0,-2(j_1+j_2)-2} &\simeq e^{-\varphi-\bar{\varphi}-i(H+\bar{H})} \Phi_{j_1+j_2;-j_1-j_2,-j_1-j_2}^{(su)} e^{Q\tilde{j}_3\phi+ip_3\cdot x}. \end{aligned} \quad (5.18)$$

The expectation value (5.17) factorizes into the contributions of the ghosts, the  $\mathbb{R}^{5,1}$  CFT and the  $\left(\frac{SU(2)}{U(1)} \times \frac{SL(2,\mathbb{R})}{U(1)}\right)/\mathbf{Z}_k$  CFT. The ghost contribution is equal to one. The  $\mathbb{R}^{5,1}$  contribution is similar to a computation performed in equation (5.9) of [3]. It gives the correct kinematic structure (compare to (5.16))<sup>19</sup>

$$\frac{8}{Q^2(j_1+\tilde{j}_1+1)^2} \xi_{\mu\nu}^{(1)} \xi_{\mu'\nu'}^{(2)} \left( p_1^\mu p_1^{\mu'} \eta^{\nu\nu'} \pm (\mu \leftrightarrow \nu, \mu' \leftrightarrow \nu') \right). \quad (5.19)$$

---

<sup>19</sup> As in [3], there is another term in this contribution proportional to  $p_1^2$ , which turns out to be analytic in all the momenta so it is interpreted as a contact-term in space-time.

What remains is the three-point function

$$\begin{aligned} & \langle e^{\frac{i}{2}(H+\bar{H}-H'-\bar{H}')} \Phi_{j_1;j_1,j_1}^{(su)} e^{Q\tilde{j}_1\phi} \cdot e^{\frac{i}{2}(H+\bar{H}+H'+\bar{H}')} \Phi_{j_2;j_2,j_2}^{(su)} e^{Q\tilde{j}_2\phi} \\ & e^{-i(H+\bar{H})} \Phi_{j_1+j_2;-j_1-j_2,-j_1-j_2}^{(su)} e^{Q\tilde{j}_3\phi} \rangle. \end{aligned} \quad (5.20)$$

The RR vertex operators that enter equation (5.20) correspond in the exact background (4.7) to (see section 4.3 of [3] for a discussion of these formulae and definitions of the operators appearing in them)

$$\begin{aligned} e^{\frac{i}{2}(H+\bar{H}-H'-\bar{H}')} \Phi_{j_1;j_1,j_1}^{(su)} e^{Q\tilde{j}_1\phi} & \leftrightarrow V_{j_1;j_1,j_1}^{(su,susy)}(RR,+) V_{\tilde{j}_1;j_1+1,j_1+1}^{(sl,susy)}(RR,-), \\ e^{\frac{i}{2}(H+\bar{H}+H'+\bar{H}')} \Phi_{j_2;j_2,j_2}^{(su)} e^{Q\tilde{j}_2\phi} & \leftrightarrow V_{j_2;j_2,j_2}^{(su,susy)}(RR,+) V_{\tilde{j}_2;j_2,j_2}^{(sl,susy)}(RR,+). \end{aligned} \quad (5.21)$$

Together with the form of the third operator in (5.20), given in (4.19), we conclude that the three-point function (5.20) is given by the following product of  $SU(2)/U(1)$  and  $SL(2, \mathbb{R})/U(1)$  three-point functions:

$$\begin{aligned} & \langle V_{j_1;j_1,j_1}^{(su,susy)}(RR,+) V_{j_2;j_2,j_2}^{(su,susy)}(RR,+) V_{\frac{k-2}{2}-j_1-j_2;\frac{k-2}{2}-j_1-j_2,\frac{k-2}{2}-j_1-j_2}^{(su,susy)} \rangle \times \\ & \langle V_{\tilde{j}_1;j_1+1,j_1+1}^{(sl,susy)}(RR,-) V_{\tilde{j}_2;j_2,j_2}^{(sl,susy)}(RR,+) V_{\tilde{j}_3;-j_1-j_2-1,-j_1-j_2-1}^{(sl,susy)} \rangle. \end{aligned} \quad (5.22)$$

The  $SU(2)/U(1)$  correlator can be simplified by using the reflection relation (see §4.3 in [3])

$$V_{j;m,m}^{(su,susy)}(RR,+) = V_{\frac{k-2}{2}-j;-\frac{k-2}{2}+m,-\frac{k-2}{2}+m}^{(su,susy)}(RR,-). \quad (5.23)$$

Applying (5.23) to (say)  $j_1$  in (5.22), we find the  $SU(2)/U(1)$  amplitude

$$\langle V_{\frac{k-2}{2}-j_1;-\frac{k-2}{2}+j_1,-\frac{k-2}{2}+j_1}^{(su,susy)}(RR,-) V_{j_2;j_2,j_2}^{(su,susy)}(RR,+) V_{\frac{k-2}{2}-j_1-j_2;\frac{k-2}{2}-j_1-j_2,\frac{k-2}{2}-j_1-j_2}^{(su,susy)} \rangle. \quad (5.24)$$

Since this is an amplitude that preserves  $U(1)$ , we can calculate it in the underlying bosonic  $SU(2)_{k-2}$  CFT, where it is given by

$$\begin{aligned} & \langle \Phi_{\frac{k-2}{2}-j_1;-\frac{k-2}{2}+j_1,-\frac{k-2}{2}+j_1}^{(su)} \Phi_{j_2;j_2,j_2}^{(su)} \Phi_{\frac{k-2}{2}-j_1-j_2;\frac{k-2}{2}-j_1-j_2,\frac{k-2}{2}-j_1-j_2}^{(su)} \rangle = \\ & \left[ \gamma\left(\frac{1}{k}\right) \gamma\left(1 - \frac{2j_1+1}{k}\right) \gamma\left(1 - \frac{2j_2+1}{k}\right) \gamma\left(\frac{2(j_1+j_2)+1}{k}\right) \right]^{\frac{1}{2}}. \end{aligned} \quad (5.25)$$

In the last step we used equation (B.11) in [3], which is taken from [33].

It remains to compute the  $SL(2, \mathbb{R})/U(1)$  three-point function on the second line of (5.22). As before, since this correlation function preserves  $U(1)$ , we can compute it in the

underlying  $SL(2, \mathbb{R})_{k+2}$  CFT. Thus, we have to compute (in notations which are described in appendix A)

$$\begin{aligned} \langle \tilde{\Phi}_{\tilde{j}_1; j_1+1, j_1+1} \tilde{\Phi}_{\tilde{j}_2; j_2, j_2} \tilde{\Phi}_{\tilde{j}_3; -j_1-j_2-1, -j_1-j_2-1} \rangle = \\ \tilde{D}(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3) \int d^2 x_1 d^2 x_2 |x_1|^{2(\tilde{j}_1+j_1+1)} |x_2|^{2(\tilde{j}_2+j_2)} |1-x_1|^{2(\tilde{j}_2-\tilde{j}_1-\tilde{j}_3-1)} \\ |1-x_2|^{2(\tilde{j}_1-\tilde{j}_2-\tilde{j}_3-1)} |x_1-x_2|^{2(\tilde{j}_3-\tilde{j}_1-\tilde{j}_2-1)}. \end{aligned} \quad (5.26)$$

This is precisely the computation described in §4.3 of [3], after performing the change of variables described in (2.59), and for our current purposes we are interested in a double-pole contribution to this expression. As  $\tilde{j}_1 \rightarrow j_1$ , (5.26) exhibits a pole coming from  $x_1 \rightarrow \infty$ ,

$$\begin{aligned} \langle \tilde{\Phi}_{\tilde{j}_1; j_1+1, j_1+1} \tilde{\Phi}_{\tilde{j}_2; j_2, j_2} \tilde{\Phi}_{\tilde{j}_3; -j_1-j_2-1, -j_1-j_2-1} \rangle \simeq \\ \frac{\pi^2}{\tilde{j}_1 - j_1} \tilde{D}(j_1, \tilde{j}_2, \tilde{j}_3) \gamma(\tilde{j}_3 - j_1 - j_2) \gamma(\tilde{j}_2 + j_2 + 1) \gamma(j_1 - \tilde{j}_2 - \tilde{j}_3). \end{aligned} \quad (5.27)$$

As  $\tilde{j}_3 \rightarrow j_1 + j_2$ ,  $\tilde{j}_2 \rightarrow j_2$ , this behaves as

$$\langle \tilde{\Phi}_{\tilde{j}_1; j_1+1, j_1+1} \tilde{\Phi}_{\tilde{j}_2; j_2, j_2} \tilde{\Phi}_{\tilde{j}_3; -j_1-j_2-1, -j_1-j_2-1} \rangle \simeq \frac{\pi^2 \tilde{D}(j_1, j_2, j_1 + j_2)}{(\tilde{j}_1 - j_1)(\tilde{j}_3 - j_1 - j_2)}. \quad (5.28)$$

Evaluating  $\tilde{D}(j_1, j_2, j_1 + j_2)$  using the formulae in appendix A we finally find

$$\begin{aligned} \langle \tilde{\Phi}_{\tilde{j}_1; j_1+1, j_1+1} \tilde{\Phi}_{\tilde{j}_2; j_2, j_2} \tilde{\Phi}_{\tilde{j}_3; -j_1-j_2-1, -j_1-j_2-1} \rangle \simeq \\ \frac{\pi^2}{(\tilde{j}_1 - j_1)(\tilde{j}_3 - j_1 - j_2)} \left[ \gamma\left(\frac{1}{k}\right) \gamma\left(1 - \frac{2j_1 + 1}{k}\right) \gamma\left(1 - \frac{2j_2 + 1}{k}\right) \gamma\left(\frac{2(j_1 + j_2) + 1}{k}\right) \right]^{-\frac{1}{2}}. \end{aligned} \quad (5.29)$$

Multiplying by the  $SU(2)/U(1)$  contribution (5.25), we find that the correlator (5.22) is given by

$$\begin{aligned} \langle V_{j_1; j_1, j_1}^{(su, susy)}(RR, +) V_{j_2; j_2, j_2}^{(su, susy)}(RR, +) V_{\frac{k-2}{2}-j_1-j_2; \frac{k-2}{2}-j_1-j_2, \frac{k-2}{2}-j_1-j_2}^{(su, susy)} \rangle \times \\ \langle V_{\tilde{j}_1; j_1+1, j_1+1}^{(sl, susy)}(RR, -) V_{\tilde{j}_2; j_2, j_2}^{(sl, susy)}(RR, +) V_{\tilde{j}_3; -j_1-j_2-1, -j_1-j_2-1}^{(sl, susy)} \rangle \simeq \\ \frac{\pi^2}{(\tilde{j}_1 - j_1)(\tilde{j}_3 - j_1 - j_2)} \simeq \frac{4\pi^2}{k^2 p_1^2 p_3^2} (2j_1 + 1) [2(j_1 + j_2) + 1]. \end{aligned} \quad (5.30)$$

We see that the string calculation (5.19), (5.30) gives rise to the correct kinematics, reproducing the pole structure of the first term in (5.16). It remains to check that the dependence on  $j_1, j_2$  comes out correctly as well.

To do that we first need to assemble all the factors in the string calculation, and then take into account the relation between the string theory vertex operators and the field theory operators. Bringing together the results (5.19), (5.30), we find that the residue of the double pole in the string theory three-point function (5.17) is proportional to

$$\langle \mathcal{O}_{2j_1+1}^+ \mathcal{O}_{2j_2+1}^+ \mathcal{O}_{0,-2(j_1+j_2)-2} \rangle \sim \frac{8k}{2(2j_1+1)^2} 4\pi^2 \frac{(2j_1+1)[2(j_1+j_2)+1]}{k^2} = \frac{16\pi^2}{k} \frac{2(j_1+j_2)+1}{2j_1+1}. \quad (5.31)$$

The relation between the vertex operator  $\mathcal{O}_{2j+1}^+$  and the corresponding gauge theory observable was determined in<sup>20</sup> equation (5.12) in [3], up to  $j$ -independent constants which were not carefully followed there,

$$\mathcal{O}_{2j+1}^+ \sim \frac{1}{(2j+1)} \xi^{\mu\nu} \widetilde{\text{tr}}(F_{\mu\nu} B^{2j+1}). \quad (5.32)$$

Together with the relation (5.1), (5.7), the string theory predicts that the gauge theory amplitude should scale with  $j_1, j_2$  like (up to  $k$ -dependent constants):

$$\frac{1}{2(j_1+j_2)+1} (2j_1+1)(2j_2+1) \frac{2(j_1+j_2)+1}{2j_1+1} = (2j_2+1). \quad (5.33)$$

Thus, we see that the dependence on  $n_1 = 2j_1+1, n_2 = 2j_2+1$  comes out correctly as well: the first term in (5.16) is indeed proportional to  $n_2$  and is independent of  $n_1$ . Thus, we find precise agreement of the behavior near the poles between the string theory and low-energy gauge theory results, consistent with the fact that the string theory contribution (5.27) comes purely from the region  $x \rightarrow 0, \infty$  and with our general discussion. Matching the constants (which we would need to do in order to determine the precise value of  $g_s^2$  appearing in (5.7)) requires more work, and we will not attempt to do this here.

$$5.5. \langle \widetilde{\text{tr}}(F_{\mu_1\nu_1} B^{n_1}) \widetilde{\text{tr}}(F_{\mu_2\nu_2} B^{n_2}) \widetilde{\text{tr}}(F_{\mu_3\nu_3} (B^*)^{n_3}) \widetilde{\text{tr}}(F_{\mu_4\nu_4} (B^*)^{n_4}) \rangle$$

The contributions to the correlation functions we discussed until now from the low-energy gauge theory all came simply from free-field contractions, and they do not teach us anything about non-trivial S-matrix elements in this theory. This is due to the particularly simple examples we have chosen, and is not a general property of our formalism. An example of a non-trivial S-matrix computation arises from the 4-point function of the

---

<sup>20</sup> Here and below we omit the powers of  $M_W$ .

operators (5.15). This 4-point function was computed (at tree-level) in [3], where it was shown that it has a quadruple pole as a function of all the momenta when  $p_\mu^2 = 0$ , and that the coefficient of this pole may be interpreted as a non-trivial S-matrix element, arising from an  $F^4$  interaction in the low-energy field theory. The result of this computation was verified by comparing it with predictions from the duality of type IIA string theory on K3 with heterotic string theory on  $T^4$ . The analysis of the present paper makes it clear that this agreement is based on the fact that the  $SL(2, \mathbb{R})/U(1)$  contribution to the residue of the pole is localized near  $x = 0, \infty$ , so (as assumed in [3]) it arises purely from the low-energy gauge theory. Other non-trivial S-matrix elements may be similarly computed by using the methods of [3] and this paper.

### 5.6. $\langle \widetilde{\text{tr}}(F_{\mu\nu} B^n) \widetilde{\text{tr}}(F_{\mu'\nu'} B^{k-n}) \rangle$

As a final example in this section let us discuss a two-point function which does not conserve winding number, involving operators of the form (5.15). As we discussed in §2.6, one way to compute a correlation function like  $\langle \mathcal{O}_n^+ \mathcal{O}_{k-n}^+ \rangle$ , which violates winding number by one unit, is to insert into the correlation function an additional degenerate vertex operator  $\Phi_{-(k+2)/2, -(k+2)/2, -(k+2)/2}$ . The resulting correlation function conserves winding number, so it can be analyzed by the same methods we used above, and as discussed in §2.6, its  $SL(2)/U(1)$  part is precisely the same as that of the two-point function that we are interested in.

Another way to compute this two-point function is to use the reflection property (2.62). As discussed in [3], this reflection property relates the normalizable part of the operator  $\mathcal{O}_n^+ \simeq \xi^{\mu\nu} \widetilde{\text{tr}}(F_{\mu\nu} B^n)$  (coming from its massless pole) to that of  $\mathcal{O}_{k-n}^- \simeq \xi^{\mu\nu} \widetilde{\text{tr}}(F_{\mu\nu} (B^*)^{k-n})$ . Thus, we can use it to relate the two-point function we are interested in to the two-point function  $\langle \mathcal{O}_{k-n}^- \mathcal{O}_{k-n}^+ \rangle$  which conserves winding number. From the point of view of the low-energy field theory this relation between the two operators  $\widetilde{\text{tr}}(F_{\mu\nu} B^n)$  and  $\widetilde{\text{tr}}(F_{\mu\nu} (B^*)^{k-n})$  (of different winding numbers) is obvious, since the one-particle states that these operators create are the same due to the relation  $\langle B^n \rangle = M_W^{2n-k} \langle (B^*)^{k-n} \rangle$  which follows from (4.4). In this way of computing, the final two-point function that we end up with is similar to the one discussed in §5.1, and again it receives contributions both from an LSZ pole (related to the low-energy field theory) and from a bulk pole.

Note that the simplest example of a winding-number non-conserving correlation function is just the one-point function of the operator  $\widetilde{\text{tr}}(B^k)$ :  $\langle \widetilde{\text{tr}}(B^k) \rangle = k M_W^k$  (4.4). This can be computed by similar methods, or by computing its derivative with respect to the  $\mathcal{N} = 2$  Liouville coupling (3.9), which gives a winding-conserving two-point function.



## 6. Fixing the mixing

In this section we discuss another class of correlation functions in the six dimensional DSLST, whose low energy behavior is expected from the analysis of §2-§4 to be dominated by the low energy gauge theory. By comparing the string calculation of these correlation functions to the gauge theory one we determine the precise form of the gauge theory operators on the right hand side of (4.11), including all single-trace and multi-trace contributions.

We wish to study correlation functions of the operators<sup>21</sup>

$$\mathcal{B}_n \equiv \frac{1}{n} \text{tr}(B^n) \simeq \mathcal{O}_{0,n} , \quad (6.1)$$

corresponding to the vertex operators (4.16), (4.19) (obeying (4.14)) in six dimensional  $\mathcal{N} = (1, 1)$  supersymmetric LST. Consider a general correlation function of these operators,

$$\langle \mathcal{B}_{n_1}(p^1) \mathcal{B}_{n_2}(p^2) \cdots \mathcal{B}_{n_r}(p^r) \bar{\mathcal{B}}_{\hat{n}_1}(\hat{p}^1) \cdots \bar{\mathcal{B}}_{\hat{n}_{\hat{r}}}(\hat{p}^{\hat{r}}) \rangle , \quad (6.2)$$

where  $\bar{\mathcal{B}}_n(p)$  is the complex conjugate of  $\mathcal{B}_n(-p)$ . As in §2, we will impose winding conservation (2.34). This leads to the constraint<sup>22</sup>  $\sum_{i=1}^r n_i = \sum_{i=1}^{\hat{r}} \hat{n}_i$ . We will also restrict to  $r + \hat{r} > 2$ ; the case of the two-point function was already discussed in §5.1.

In order to understand the structure of (6.2) at low momenta, consider the behavior of the correlation function

$$\langle \text{tr}(B^{n_1})(p^1) \cdots \text{tr}(B^{n_r})(p^r) \text{tr}(B^{*\hat{n}_1})(\hat{p}^1) \cdots \text{tr}(B^{*\hat{n}_{\hat{r}}})(\hat{p}^{\hat{r}}) \rangle \quad (6.3)$$

in the low energy field theory. The leading contribution to this correlation function is due to standard free field contractions in which  $B$ 's from the operators  $\text{tr}(B^{n_i})$  are contracted with  $B^*$ 's from the operators  $\text{tr}(B^{*\hat{n}_i})$ . The minimal number of contractions needed to get a connected diagram is  $r + \hat{r} - 1$ . Replacing the uncontracted  $B$ 's and  $B^*$ 's by their expectation value (4.4), we find that the correlation function (6.3) behaves at large  $M_W$  like  $M_W^x$  with

$$x = \sum_{i=1}^r n_i + \sum_{i=1}^{\hat{r}} \hat{n}_i - 2(r + \hat{r} - 1) = 2 \left[ \sum_{i=1}^r n_i - (r + \hat{r} - 1) \right] . \quad (6.4)$$

---

<sup>21</sup> The overall normalization of these operators, which can be determined as discussed in §5.1, will not be important in this section.

<sup>22</sup> Note that the  $\mathbf{Z}_k$  symmetry  $B \rightarrow e^{\frac{2\pi i}{k}} B$ , which is preserved (up to a gauge transformation) by the background (4.4), implies that in general  $\sum_{i=1}^r n_i - \sum_{i=1}^{\hat{r}} \hat{n}_i \in k\mathbf{Z}$ .

Terms with more than  $r + \hat{r} - 1$  contractions scale like lower powers of  $M_W$  and hence are subleading in the  $1/M_W$  expansion. Recalling that in string theory on (4.7) the  $1/M_W$  expansion is equivalent to the string loop expansion (see (4.8)), we expect these terms to come from higher loop diagrams in string theory.

Why is there no contribution to the correlation function (6.3) from higher order terms in the Lagrangian of the six dimensional gauge theory? As mentioned above, since this theory is non-renormalizable, we must allow interaction terms in the Lagrangian that go like arbitrarily high powers of the fields. For example, an interaction of the form  $\text{tr}(B^n B^{*n})$  with  $n \geq r, \hat{r}$  would contribute to the correlation function (6.3) terms that go like a power of  $M_W$  that increases linearly with  $n$ .

The answer to this question is that we know that the eigenvalues of  $B$  are exact moduli in the full theory on the fivebranes, and thus the Lagrangian for  $B$  must have the symmetry  $B \rightarrow B + C$ , with  $C$  an arbitrary diagonal matrix. Taking this into account, it is easy to convince oneself that interactions of this form cannot exist, and the free contractions indeed give the leading contribution in the  $1/M_W$  expansion.

A useful special case is  $\hat{r} = 1$ , for which one has

$$\langle \text{tr}(B^{n_1})(p^1) \cdots \text{tr}(B^{n_r})(p^r) \text{tr}(B^{*n})(\hat{p}) \rangle \simeq M_W^{2(n-r)} \prod_{i=1}^r \frac{n_i}{(p^i)^2}, \quad (6.5)$$

where  $\sum_{i=1}^r n_i = n$ . In this case, the leading contribution comes from contracting one  $B$  from each of the first  $r$  operators with a  $B^*$  from the last operator. This leads to the structure indicated on the right hand side of (6.5). Note that there are poles corresponding to the first  $r$  momenta, but not to the last one. This is very natural, since if there was a pole associated with the last external leg as well, the S-matrix for scattering  $B$  particles would be non-zero at zero momentum, which would be inconsistent with the expected absence of a potential for the eigenvalues of  $B$ .

Equations (6.3), (6.5) are field theory correlation functions of single trace operators. For comparison with string theory we have to generalize the discussion to more general operators,

$$\text{tr}(B^n) \rightarrow \tilde{\text{tr}}(B^n) = \text{tr}(B^n) + \sum_{l_1+l_2=n} c_{l_1,l_2} \text{tr}(B^{l_1}) \text{tr}(B^{l_2}) + \cdots. \quad (6.6)$$

The multi-trace terms in (6.6) contribute to some of the amplitudes (6.3) at various orders in the  $1/M_W$  expansion. Consider for example the special case (6.5). It is easy to see

that at the order in  $M_W$  indicated on the right hand side, multi-trace contributions to the first  $r$  operators,  $\text{tr}(B^{n_i})$ , vanish. The reason is that there can only be one free field contraction that involves a given operator  $\text{tr}(B^n)$ . Thus, contributions of multi-trace operators necessarily include factors of  $\langle \text{tr}(B^{l_i}) \rangle$ , which vanish for all  $l_i < k$  at the point in moduli space at which we are working, (4.4).

Multi-trace contributions to the last operator,  $\text{tr}(B^{*n})$ , *do* contribute to (6.5). For example, terms of the form  $\text{tr}(B^{*n_1})\text{tr}(B^{*n_2}) \cdots \text{tr}(B^{*n_r})$ ,  $\text{tr}(B^{*n_1+n_2})\text{tr}(B^{*n_3}) \cdots \text{tr}(B^{*n_r})$ ,  $\text{tr}(B^{*n_1+n_2+\cdots+n_{r-1}})\text{tr}(B^{*n_r})$ , etc, all contribute at the same order as  $\text{tr}(B^{*n})$ , and give rise to the same analytic structure as that on the right hand side of (6.5). This fact will prove useful below.

Having understood the field theory amplitudes (6.2) – (6.5), we now turn to the string theory calculation of the corresponding correlation function of the vertex operators (4.19). Our first task is to reproduce the scaling of the correlation function with  $M_W$ . To do that, note that the vertex operator  $\mathcal{B}_n$  (4.16) goes for small  $p_\mu^2$  as  $e^{Q(n-2)\phi/2}$ , due to the physical state condition (4.14). The sphere correlation function (6.2) scales with the  $\mathcal{N} = 2$  Liouville coupling  $\mu$  (3.9) like  $\mu^y$ , with

$$\frac{Q}{2}(n_1 - 2 + n_2 - 2 + \cdots + n_r - 2 + \hat{n}_1 - 2 + \cdots + \hat{n}_{\hat{r}} - 2) - \frac{1}{Q}y = -Q, \quad (6.7)$$

or  $y = (\sum_i n_i + \sum_i \hat{n}_i - 2r - 2\hat{r} + 2)/k$ . Using the relation between  $\mu$  and  $M_W$  (4.10) we conclude that the correlator scales like  $M_W^x$  with  $x = ky$ . Comparing to the field theory result (6.4), we see that the string and field calculations give the same answer for  $x$ .

Our next task is to calculate the string theory correlation function (6.2). In general, this correlation function is complicated; however, the case  $\hat{r} = 1$ , in which (6.2) reduces to

$$\langle \mathcal{B}_{n_1}(p^1) \mathcal{B}_{n_2}(p^2) \cdots \mathcal{B}_{n_r}(p^r) \bar{\mathcal{B}}_n(\hat{p}) \rangle \quad \text{with} \quad \sum_{i=1}^r n_i = n, \quad (6.8)$$

turns out to be much more tractable, and we will restrict to it here.

To calculate (6.8) we recall an important feature of the vertex operators  $\mathcal{B}_n$  (4.19): their  $SU(2)/U(1)$  component is chiral. Indeed, from (4.18) we see that any operator of the form  $V_{m;-m,-m}^{(su,susy)}$  has the property that its dimension and R-charge are related:  $\Delta = \bar{\Delta} = \frac{1}{2}R = \frac{1}{2}\bar{R} = m/k$ . Thus, it is a chiral operator (annihilated by  $G_{-1/2}^+$  and  $\bar{G}_{-1/2}^+$ ). Defining  $V_{\frac{1}{2};-\frac{1}{2},-\frac{1}{2}}^{(su,susy)} \equiv \chi$ , one has  $V_{m;-m,-m}^{(su,susy)} = \chi^{2m}$ . One can think of  $\chi$  as the

bottom component of a Landau-Ginzburg superfield, in terms of which the  $\mathcal{N} = 2$  minimal model is naturally formulated. In terms of  $\chi$ , the vertex operator (4.19) can be written as

$$\mathcal{O}_{0,2j+2}(p_\mu) = e^{-\varphi-\bar{\varphi}} \chi^{k-2j-2} V_{\tilde{j};j+1,j+1}^{(sl,susy)} e^{ip \cdot x} \leftrightarrow \mathcal{B}_{2j+2} . \quad (6.9)$$

To compute the  $(r+1)$ -point function (6.8) in tree-level string theory we need to put two of the vertex operators in the  $(-1, -1)$  picture as in (6.9), and the other  $(r-1)$  in the  $(0, 0)$  picture, where they are given by

$$\mathcal{B}_{2j+2}(p_\mu) \leftrightarrow G_{-1/2} \bar{G}_{-1/2} \left( \chi^{k-2j-2} V_{\tilde{j};j+1,j+1}^{(sl,susy)} e^{ip \cdot x} \right) . \quad (6.10)$$

Each of the superconformal generators  $G$  and  $\bar{G}$  can be written as a sum of contributions from the  $SU(2)/U(1)$ ,  $SL(2, \mathbb{R})/U(1)$  and  $\mathbb{R}^{5,1}$  CFTs. The three component CFTs are  $\mathcal{N} = (2, 2)$  worldsheet supersymmetric; therefore,  $G_{-1/2}$  may be further decomposed in terms of the two  $\mathcal{N} = 2$  superconformal generators as  $G_{-1/2} = G_{-1/2}^+ + G_{-1/2}^-$ , where  $G^+$  ( $G^-$ ) raises (lowers) the  $U(1)_R$ -charge by one unit. Similar comments apply to the other worldsheet chirality.

The result of the calculation cannot depend on which two operators we take to be in the  $(-1, -1)$  picture. A convenient choice is to pick the first two operators (say) in (6.8). We will next show that the correlation function (6.8) vanishes for this choice, point by point in the worldsheet moduli space.

To see that, focus on the  $SU(2)/U(1)$  part of the unintegrated correlation function (6.8). In order to get a non-vanishing result, the total  $U(1)_R$  charge must vanish. The first two operators have  $U(1)_R$  charges  $(1 - \frac{n_1}{k})$  and  $(1 - \frac{n_2}{k})$ . The next  $(r-2)$  operators have components with two different values of the R-charge. Contributions to  $G_{-\frac{1}{2}}$  in (6.10) from  $\mathbb{R}^{5,1} \times SL(2)/U(1)$  give operators with  $SU(2)/U(1)$  R-charge  $(1 - \frac{n_i}{k})$ . In the contribution to  $G_{-\frac{1}{2}}$  of the  $SU(2)/U(1)$  CFT, only the  $G^-$  part acts (recall that  $G^+$  annihilates the chiral operators  $\chi^l$ ), and gives an operator with R-charge  $(-\frac{n_i}{k})$ . Similarly, for the last operator in (6.8),  $\bar{\mathcal{B}}_n$ , we find contributions with R-charge  $(\frac{n}{k} - 1)$  or  $(\frac{n}{k})$ .

Remembering that  $n = \sum_i n_i$ , it is easy to see that all possible contributions to the correlation function (6.8) have a total R-charge which is a positive integer. Since the total R-charge does not vanish, we conclude that this correlation function vanishes in string theory for any  $r \geq 2$ . This result is true for the unintegrated correlation function of the vertex operators in (6.8); therefore, it obviously holds after integrating over the moduli as well.

Two comments:

- (1) The derivation above is valid for arbitrary momenta  $(p^1, \dots, p^r, \hat{p})$ . Below we will see that at low momenta, this vanishing has interesting consequences for the gauge-string correspondence. It would be interesting to understand the significance and implications of the vanishing of (6.8) more generally.
- (2) If both  $r$  and  $\hat{r}$  in (6.2) are larger than one, the argument above does not imply the vanishing of the correlation function, but it leads to a simplification of its structure. It would be interesting to explore this further.

The vanishing of (6.8) derived from string theory is quite surprising from the point of view of the low-energy field theory, where the free field theory diagrams do not seem to vanish. Indeed, if the operators  $\mathcal{B}_n$  were precisely identified with  $\frac{1}{n}\text{tr}(B^n)$  in the low-energy field theory we would get a contradiction, since the relevant field theory correlation function (6.5) does not vanish. One might ask whether it is possible that this is a case where non-gauge theoretic contributions to the amplitude are important, but it is not difficult to see that the residue of the poles in (6.5) should not receive bulk contributions. Hence, the vanishing of (6.8) should have an interpretation purely in the low energy gauge theory.

The resolution of the apparent paradox is that in general the LST operators  $\mathcal{B}_n$  corresponding to the single string vertex operators (6.9) reduce at low energies to linear combinations of single-trace and multi-trace operators<sup>23</sup>

$$\mathcal{B}_n \simeq \frac{1}{n}\text{tr}(B^n) + \sum_{n_1} a_{n_1} \text{tr}(B^{n_1})\text{tr}(B^{n-n_1}) + \sum_{n_1, n_2} a_{n_1, n_2} \text{tr}(B^{n_1})\text{tr}(B^{n_2})\text{tr}(B^{n-n_1-n_2}) + \dots \quad (6.11)$$

with some coefficients  $a$ . As mentioned above, at the particular point in moduli space (4.4) that we are studying, where  $\langle \text{tr}(B^l) \rangle = 0$  for all  $l < k$ , only the first term in the expansion

---

<sup>23</sup> It is important to distinguish this mixing from the mixings between single-trace and multi-trace operators computed in the context of the AdS/CFT correspondence. The mixings computed there usually involve non-chiral operators, for which specific combinations of the single-trace and multi-trace operators are eigenfunctions of the dilatation operator. On the other hand, for chiral operators there is no intrinsic preference for a particular linear combination over any other combination from the field theory point of view. However, it is still true in all examples of the AdS/CFT correspondence that the string theory vertex operators map to specific linear combinations of operators. It would be interesting to work out the generalization of our results to other examples. Of course, generally this is difficult due to the presence of RR backgrounds, but it should be possible to do this at least in the BMN limit [34] of string theory on  $AdS_5 \times S^5$ .

of the  $\mathcal{B}_{n_i}$  operators in (6.8) can contribute (at leading order in  $M_W$ ). However, many different terms in the expansion (6.11) of the operator  $\bar{\mathcal{B}}_n$  can contribute. The correlation function (6.8) is a linear combination of all of these contributions, and its vanishing may be used to determine the value of the relevant coefficients in (6.11). By looking at all correlation functions of the type (6.8) we can determine all the coefficients appearing in (6.11); this computation is detailed in appendix B. The result of appendix B may be written as<sup>24</sup>:

$$\mathcal{B}_n \simeq \sum_{l=1}^{\infty} \sum_{n_i=2}^n \frac{1}{l!} \left( \frac{1-n}{k} \right)^{l-1} \delta\left(\sum_i n_i, n\right) \left( \frac{1}{n_1} \text{tr}(B^{n_1}) \right) \left( \frac{1}{n_2} \text{tr}(B^{n_2}) \right) \cdots \left( \frac{1}{n_l} \text{tr}(B^{n_l}) \right) \quad (6.12)$$

for  $n = 2, 3, \dots, k$ . Note that the sum over the  $n_i$  in (6.12) goes over all possible values of the  $n_i$  without any ordering between them, so that (for instance) a term involving  $l$  specific traces with all the  $n_i$  different from each other will appear  $l!$  times in the sum, canceling the factor of  $1/l!$  appearing explicitly in the formula.

We can test whether the result (6.12) is reasonable by considering two different limits. In a general  $SU(k)$  gauge theory with fields in the adjoint representation, 't Hooft has argued [35] that in the limit of large  $k$  with fixed  $g_{YM}^2 k$  the gauge theory may be described as a string theory, with single-trace operators (involving a product of a finite number of adjoint fields) mapping to string vertex operators. We expect that the same should be true also in the LST (even though it is not simply a gauge theory), namely that for large  $k$  and finite  $n$  we should have  $\mathcal{B}_n \simeq \frac{1}{n} \text{tr}(B^n)$ . And indeed, in this limit only the  $l = 1$  term in (6.12) contributes, and the other terms are suppressed by powers of  $1/k$ , as expected. Note that we have not taken any large  $k$  limit in deriving (6.12) (or elsewhere in our computations).

The opposite limit of large  $n$  and  $k$  with small  $(k - n)$  corresponds to operators with a large angular momentum on the  $S^3$  (in the asymptotic region (4.1)). It was argued in [36] that in AdS/CFT examples such operators are not described by perturbative string states but rather by “giant graviton” wrapped D-brane states, since the Ramond-Ramond (RR) flux in these backgrounds causes states with large angular momentum to expand into D-branes. It was further argued in [37-41] (by examining various correlation functions of

---

<sup>24</sup> In appendix B we prove this formula using a certain plausible assumption. A complete proof is still missing.

such states) that these states should be identified with subdeterminant operators of the form

$$\text{subdet}_n(B) \equiv \frac{(-1)^n}{n!(k-n)!} \epsilon^{a_1 a_2 \dots a_n a_{n+1} \dots a_k} \epsilon_{b_1 b_2 \dots b_n a_{n+1} \dots a_k} B_{a_1}^{b_1} B_{a_2}^{b_2} \dots B_{a_n}^{b_n} \quad (6.13)$$

rather than with single-trace states<sup>25</sup>. In our theory we do not have an RR flux but we have an NS-NS three-form flux on the  $S^3$ , so similar arguments suggest that states with large angular momentum should again correspond to “giant gravitons,” but now these “giant gravitons” are fundamental strings wrapped on an  $S^1$  inside the  $S^3$  rather than D-branes, explaining why we are able to describe them (and see the bound on their angular momentum) in string perturbation theory. Most of the arguments that the “giant gravitons” should map to subdeterminants may also be generalized to our case, so we expect that for  $k - n \ll k$  the operators (6.12) should be approximately equal to the operators (6.13).

This is indeed true, since the operators (6.13) may be written as

$$\text{subdet}_n(B) = \sum_{l=1}^{\infty} \sum_{n_i=2}^n \frac{(-1)^l}{l!} \delta\left(\sum_i n_i, n\right) \left(\frac{1}{n_1} \text{tr}(B^{n_1})\right) \left(\frac{1}{n_2} \text{tr}(B^{n_2})\right) \dots \left(\frac{1}{n_l} \text{tr}(B^{n_l})\right); \quad (6.14)$$

this formula<sup>26</sup> may be easily seen to be a solution to the recursion relation

$$n \cdot \text{subdet}_n(B) + \sum_{i=2}^n \text{subdet}_{n-i}(B) \cdot \text{tr}(B^i) = 0 \quad (6.15)$$

which the subdeterminant operators obey (with  $\text{subdet}_0(B) = 1$ ,  $\text{subdet}_1(B) = 0$ ). The expression for the subdeterminant (6.14) is almost the same as the formula (6.12) describing the dual of single string vertex operators, except for an (unimportant) overall sign, and the factor of  $((n-1)/k)^{l-1}$  in (6.12). However, for large  $k$  and small  $k-n$  this factor goes to

---

<sup>25</sup> The subdeterminant operators may be written in terms of the eigenvalues  $\hat{b}_i$  ( $i = 1, \dots, k$ ) of the matrix  $B$  as  $\text{subdet}_n(B) = (-1)^n \sum_{i_1 < i_2 < \dots < i_n} \hat{b}_{i_1} \hat{b}_{i_2} \dots \hat{b}_{i_n}$ .

<sup>26</sup> We have not found the formula (6.14) for subdeterminants in the literature, but at least for the case  $n = k$  it is easy to show that it is equivalent to existing formulae for the determinant. For a general matrix whose trace does not necessarily vanish one has the same formula with  $n_i \geq 1$ .

one for any finite  $l$ , so the operators we find indeed approach<sup>27</sup> subdeterminants as  $n \rightarrow k$ , though they are never precisely equal to them. The formula (6.12) provides (for large  $k$ ) a smooth interpolation between the two limits where the operators look like single traces and like subdeterminants, and based on the arguments above it is exact for all values of  $n$  and  $k$ .

As mentioned above, our argument for the vanishing of (6.8) does not apply to general correlation functions of the form (6.2) with  $r, \hat{r} > 1$ , for which one has contributions to the correlation function with vanishing total R-charge in  $SU(2)/U(1)$ . Indeed, using (6.12) one can compute the low-energy limit of such correlation functions and find that it is not vanishing. It would be interesting to verify that the non-zero tree-level correlation functions of the form (6.2) are consistent with (6.12) at low energies.

In (6.12) we identified precisely the low-energy field theory content of the operators  $\mathcal{B}_n$ , by using a computation performed at a particular point in the moduli space of the LST. However, since this identification involves non-normalizable operators it should be independent of the moduli. The operators  $\mathcal{B}_n$  are related by the global  $SO(4)$  symmetry of the LST (at the origin of its moduli space) and by supersymmetry to all chiral operators in the LST, so from (6.12) we can read off the precise low-energy field theory content of all of these chiral operators. For example, for the operators  $\mathcal{F}_n = \tilde{\text{tr}}(F_{\mu\nu} B^{n-1})$  discussed in [3] we find<sup>28</sup>

$$\mathcal{F}_n \simeq \sum_{l=1}^{\infty} \sum_{n_i=2}^n \frac{\delta(\sum_i n_i, n)}{(l-1)!} \left( \frac{1-n}{k} \right)^{l-1} (\text{tr}(F_{\mu\nu} B^{n_1-1})) \left( \frac{\text{tr}(B^{n_2})}{n_2} \right) \cdots \left( \frac{\text{tr}(B^{n_l})}{n_l} \right), \quad (6.16)$$

and for the operators  $\mathcal{O}_{1,n-1} = \tilde{\text{tr}}(AB^{n-1})$  we find

$$\mathcal{O}_{1,n-1} \simeq \sum_{l=1}^{\infty} \sum_{n_i=2}^n \frac{\delta(\sum_i n_i, n)}{(l-1)!} \left( \frac{1-n}{k} \right)^{l-1} (\text{tr}(AB^{n_1-1})) \left( \frac{\text{tr}(B^{n_2})}{n_2} \right) \cdots \left( \frac{\text{tr}(B^{n_l})}{n_l} \right). \quad (6.17)$$

---

<sup>27</sup> More precisely, the coefficients of the terms in (6.12) with a finite number of traces for large  $k$  approach those of the subdeterminant operators. This is not true for the coefficients of terms with a large number of traces (of order  $k$ ). However, this difference between the operators does not contribute for large  $k$  to the correlation functions of “giant gravitons” that motivate their identification with subdeterminants, so it is not inconsistent. It is amusing that the two formulae coincide exactly for  $n = k + 1$ , outside the range in which the string theory operators exist, where both expressions give operators which vanish (at least classically).

<sup>28</sup> The chiral operator  $\mathcal{F}_n$  includes also fermionic contributions, which we omit here.



We can use equation (6.17) to perform another consistency check of our analysis. As we saw before (see *e.g.* equation (4.15)), the left-moving component of the operator  $\mathcal{O}_{1,n-1}$  is identical to that of the operator  $\mathcal{O}_{0,n} \sim \mathcal{B}_n$ ; in particular, it also involves (in the  $(-1, -1)$  picture) a chiral primary operator in the  $SU(2)/U(1)$  CFT of R-charge  $(1 - \frac{n}{k})$ . Consider the correlation function

$$\langle \mathcal{O}_{1,n_1-1}(p^1) \mathcal{B}_{n_2}(p^2) \cdots \mathcal{B}_{n_r}(p^r) \bar{\mathcal{O}}_{1,n-1}(\hat{p}) \rangle \quad \text{with} \quad \sum_{i=1}^r n_i = n. \quad (6.18)$$

As far as the left-movers are concerned, it is identical to (6.8). Thus, we can repeat the arguments used in proving that (6.8) vanishes to show that (6.18) vanishes (at leading order in  $1/M_W$ , corresponding to string tree-level) as well.

In the low-energy gauge theory, there are various contributions to this correlation function, involving (at leading order in  $1/M_W$ ) contracting the  $A$  from the first operator with the  $A^*$  from the last operator, and performing the other contractions as before. Using (6.12) and (6.17) we can compute the coefficients of all of these different contributions and sum them up. We find that the result is zero, consistent with the string theory computation of (6.18). This provides another consistency check of our analysis.

## 7. Four dimensional $\mathcal{N} = 2$ supersymmetric LST

Our discussion above focussed on the case of  $d = 6$ , but the same considerations apply to all other LSTs. As an additional example we discuss here the case of  $d = 4$  LSTs with  $\mathcal{N} = 2$  supersymmetry which arise as decoupled theories on generalized conifold singularities of the form

$$z_1^n + z_2^2 + z_3^2 + z_4^2 = \mu \quad (7.1)$$

in type II string theory (or alternatively on wrapped  $NS5$ -branes). The low-energy theory in this case is a  $d = 4$   $\mathcal{N} = 2$   $U(1)^{n-1}$  gauge theory, which can be thought of as describing an  $SU(n)$  gauge theory near an Argyres-Douglas type point [42-44] in its moduli space (the point  $\mu = 0$ , which corresponds to a non-trivial superconformal field theory for  $n > 2$ ). Denoting the complex scalar field in the  $SU(n)$  gauge multiplet by  $\Phi$ , the gauge-invariant chiral operators are  $\text{tr}(\Phi^l)$  ( $l = 2, 3, \dots, n$ ) and their descendants. We can choose a basis of operators such that their vacuum expectation values vanish for  $\mu = 0$ , and moving away from this point is achieved by giving a non-zero expectation value to  $\Phi$  (which plays the same role as  $B$  in our discussion of the  $d = 6$  case above).

The worldsheet description of this decoupled theory is given by an orbifold of the supersymmetric  $\mathbb{R}^{3,1} \times SL(2)/U(1) \times SU(2)_n/U(1)$  CFT, similar to (4.7), where the  $SL(2)/U(1)$  theory has  $Q^2 = 1 + 2/n$ . Using the same notations as in §4, the vertex operators for the chiral operators on the worldsheet take the form [12]

$$\mathcal{O}_i(p_\mu) = e^{-\varphi - \bar{\varphi}} V_{i/2; -i/2, -i/2}^{(su, susy)} V_{j; m, m}^{(sl, susy)} e^{ip_\mu x^\mu} \quad (7.2)$$

with  $m = (n - i)/(n + 2)$ , for  $i = 0, 1, \dots, n - 2$ . The mass-shell condition for this operator is

$$p^2 = \frac{(i + 2)(n - i) + j(j + 1)(n + 2)^2}{n(n + 2)}. \quad (7.3)$$

Our analysis in §2, §3 suggests that for the values of  $i$  which give  $1 > m > 1/2$ , namely  $i < n/2 - 1$ , the operator (7.2) has an LSZ pole for  $j = m - 1$ . Using (7.3), we find that this pole is at  $p^2 = 0$ . On the other hand, for  $i > n/2 - 1$  we find no LSZ poles<sup>29</sup>. In particular, the appropriate solution of the mass-shell condition for these operators when  $p^2 = 0$  is  $j = -m > -1/2$ , and there is no LSZ pole at this value of  $j$ . Note that (unlike in  $d = 6$  LST) in the two-point-functions of  $\mathcal{O}_i$  (7.2) there is no bulk contribution to the low energy poles.

The analytic structure we found in string theory at small momenta agrees precisely with the mapping of these operators to the low-energy field theory which was suggested in [12]. The operators with  $i < n/2 - 1$  were argued to correspond to bottom components of space-time anti-chiral superfields, of the form  $\widetilde{\text{tr}}((\Phi^*)^{n-i})$ . Since  $\Phi$  has a non-zero VEV, these operators are expected to exhibit massless LSZ poles, as we found above. On the other hand, the operators with  $i > n/2 - 1$  were argued to correspond to the top components of superfields whose bottom component is  $\widetilde{\text{tr}}(\Phi^{i+2})$ . These operators do not include any component of the form  $\widetilde{\text{tr}}(X\Phi^l)$  (for a field  $X$  whose two-point function has a pole; they include, for instance, components of the form  $\widetilde{\text{tr}}(\partial_\mu \Phi \partial^\mu \Phi \Phi^i)$ ), so they are not expected to exhibit massless poles, in agreement with our string theory results.

A similar analysis may be performed for RR operators in this case. These map in space-time to operators of the form  $\widetilde{\text{tr}}(F_{\mu\nu} \Phi^l)$ , and again we find precise agreement between the string theory results and the low-energy field theory expectations. By computing the S-matrix related to these operators we can reconstruct the  $F^{2n}$  terms in the effective action of this LST, as in [3].

---

<sup>29</sup> The case  $i = n/2 - 1$  ( $m = 1/2$ ), which can occur for even values of  $n$ , is a special case for which the mass gap to the bulk continuum goes to zero, so we cannot tell whether a massless pole exists in this case or not.

## 8. Two dimensional string theory

In this paper we focused on spacetimes of the form (1.4), but much of the discussion is more general, and should apply to other asymptotically linear dilaton backgrounds in string theory. A famous example of such a background is Liouville theory, and in particular its application to two dimensional string theory (for reviews see *e.g.* [45,17,46]). In this section we briefly discuss two dimensional string theory from the point of view of our analysis.

Consider string theory in a two dimensional Euclidean space labeled by the coordinates  $(\phi, x)$ .  $\phi$  is a Liouville coordinate, which is described asymptotically by a free field with a linear dilaton (as in (1.1)) whose slope  $Q$  is related to its central charge by the relation<sup>30</sup>  $c_L = 1 + 6Q^2$ . The strong coupling singularity at  $\phi \rightarrow \infty$  is resolved by adding to the worldsheet Lagrangian the cosmological term

$$\delta\mathcal{L} = \mu_0 e^{2b\phi}, \quad (8.1)$$

where  $b$  is related to  $Q$  by  $Q = b^{-1} + b$ . This interaction creates a wall repelling the worldsheet fields from the strong coupling region  $\phi \rightarrow \infty$  and thus regularizes the theory.

The worldsheet cosmological constant  $\mu_0$  plays in this theory a role similar to that of the Sine-Liouville coupling (2.48) and the  $\mathcal{N} = 2$  Liouville coupling (3.9) in the  $SL(2, \mathbb{R})/U(1)$  backgrounds studied earlier in this paper. In particular, one can again think of  $\mu_0$  as setting an energy scale (an analog of  $M_W$  (4.10) in our previous discussion) and of string perturbation theory as an expansion in powers of the inverse of this scale.

A set of natural observables in Liouville theory is

$$V_\alpha = e^{2\alpha\phi}. \quad (8.2)$$

The scaling dimension of  $V_\alpha$  is  $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q - \alpha)$ . These operators play a role similar to that of the  $SL(2, \mathbb{R})/U(1)$  observables  $V_{j;m,\bar{m}}$  (2.3), (2.4), and their supersymmetric analogs (3.3). We saw that the observables on the cigar satisfy a reflection relation, (2.7), which played an important role in our discussion. In particular, singularities of the reflection coefficient  $R$  provided important information on the possible singularities of correlation

---

<sup>30</sup> In this section we are using conventions that are different from those of the previous sections, but are standard in studies of Liouville theory and two dimensional string theory. In particular, we take  $\alpha' = 1$  and take  $\phi \rightarrow -\phi$ , such that the weakly coupled region is at  $\phi \rightarrow -\infty$ .

functions in that case. The Liouville observables satisfy a similar reflection property (see *e.g.* [47]):

$$\begin{aligned} V_\alpha &= R_\alpha V_{Q-\alpha}, \\ R_\alpha &= -\mu^{\frac{Q-2\alpha}{b}} \frac{\Gamma(-\frac{1}{b}(Q-2\alpha))\Gamma(-b(Q-2\alpha))}{\Gamma(\frac{1}{b}(Q-2\alpha))\Gamma(b(Q-2\alpha))}, \end{aligned} \quad (8.3)$$

where  $\mu = \pi\mu_0\gamma(b^2)$ .

We are interested here in the case  $b = 1$ ,  $Q = 2$ , for which the Liouville central charge is  $c_L = 25$ . Together with the second space-like coordinate  $x$ , we have  $c = 26$ , so we can study bosonic string theory on this space.

Bosonic string theory in the two dimensional Euclidean space labeled by  $(\phi, x)$  has one field theoretic degree of freedom, the massless “tachyon” whose vertex operator is given by

$$T_k = e^{ikx + (2-|k|)\phi}. \quad (8.4)$$

In addition to the tachyon, the theory has some discrete states that occur only at some particular values of the momenta,  $k \in \mathbf{Z}$ .

The dynamics of the tachyon is rather well understood, both in the continuum formulation, and in the dual matrix quantum mechanics in the double scaling limit. The operators (8.4) are non-normalizable observables in the theory, and we can compute their correlation functions. One finds that these correlation functions have the following structure:

$$\langle T_{k_1} \cdots T_{k_n} \rangle = \left[ \prod_{i=1}^n \gamma(1 - |k_i|) \right] F(k_1, \cdots, k_n), \quad (8.5)$$

where the term in square brackets is usually referred to as the “leg pole” contribution, and  $F(k_1, \cdots, k_n)$  does not have poles as a function of the  $k_i$ .  $F$  is a continuous function of the  $k_i$ , which is bounded for finite values of the momenta; its singularities are discontinuities of the first derivative with respect to the  $k_i$ , arising from structures like  $|\sum_i k_i|$  with the sum running over some subset of the momenta.

The discrete states have always been more mysterious; in particular, correlation functions involving discrete states seemed to be divergent since they were related to those of tachyons with the special momenta  $k \in \mathbf{Z}$ , which are infinite according to (8.5).

The discussion of our paper is useful for elucidating the analytic structure of the tachyon correlation functions (8.5), and for studying the role of the discrete states. Consider first the reflection relation for the special case of two dimensional string theory:

$$V_\alpha = -\mu^{2(1-\alpha)} \frac{\Gamma^2(-2(1-\alpha))}{\Gamma^2(2(1-\alpha))} V_{2-\alpha}. \quad (8.6)$$

The reflection relation exhibits second order poles for  $2(1 - \alpha) \in \mathbf{Z}_+$ . Comparing to (8.4), we see that these double poles occur at momenta corresponding to discrete states,  $k \in \mathbf{Z}$ , and moreover the power of  $\mu$  on the right hand side of (8.6) is a positive integer in this case.

In §2 we saw that poles of the reflection coefficient have one of two origins. They either signal the presence of normalizable states living in the strong coupling region, or are due to bulk interactions that occur very far from the wall. Each of these gives rise to first order poles at the appropriate locations. In our case we find that the reflection coefficient has double poles for integer momenta, so both of those mechanisms must be operating at the same time. It must be that for  $k \in \mathbf{Z}$  there are normalizable states bound to the Liouville wall, and we see from the integer power of  $\mu$  appearing in (8.6) that bulk interactions are possible in that case as well. Each of these effects gives one of the two poles for  $k \in \mathbf{Z}$  in (8.6).

This picture is compatible with the structure of the  $n$  point functions of tachyons (8.5), which exhibit single poles as a function of each external momentum  $k_i$ . These “leg poles” are nothing but LSZ poles for the particular case of Liouville theory. For  $k \in \mathbf{Z}$  the non-normalizable tachyons  $T_k$  do not exist; one has to study instead the normalizable operators obtained by taking the residues of the LSZ poles, as in the general discussion of §2 (see *e.g.* (2.13)).

There is also an analog of the discussion at the end of §2.4 of the question whether the same poles can sometimes be thought of as bulk poles and sometimes as LSZ poles. The leg poles in (8.5) were first found in bulk correlation functions [48,25] where one takes  $n - 1$  of the momenta to be positive (say) and one negative. In that case, it was shown in [48,25] that the  $n - 1$  poles corresponding to the positive momentum tachyons are LSZ poles, while the last pole, corresponding to the negative momentum tachyon, was a bulk pole. As we saw in §2.4, it is a rather general phenomenon that the same poles have seemingly different interpretations in different ways of doing the calculation. We expect that the situation should be the same here, namely, for generic kinematics there should be a way to do the calculation in such a way that it is manifest that all  $n$  leg poles in (8.5) are LSZ poles.

Finally, the discussion above clarifies the status of the discrete states in two dimensional string theory. Unlike tachyons with generic (non-integer) momenta, there are no

non-normalizable vertex operators corresponding to these observables. They must be described by normalizable vertex operators, which can be formally defined by the procedure of going to the poles outlined in §2.

It is also useful to note that the discussion of this section generalizes in a simple way to the case of two dimensional type 0 string theory, since the structure of reflection coefficients and correlation functions in that theory is almost identical to the bosonic case.

## 9. The Thermodynamics of Little String Theories

In this section we use our improved understanding of the physics of normalizable states on the cigar to study the thermodynamics of LSTs, and in particular the possible existence of a Hagedorn phase transition in these theories.

The thermodynamics of LSTs was studied in [49-57]. In the bulk description, LST at a large energy density (in string units) is described by the near-horizon limit of near-extremal  $NS5$ -branes (or singularities), in which the background (1.3) is replaced by<sup>31</sup>

$$SL(2, \mathbb{R})_k/U(1) \times \mathbb{R}^{d-1} \times \mathcal{M}. \quad (9.1)$$

For the theory at finite energy density, the  $SL(2, \mathbb{R})/U(1)$  in (9.1) is Lorentzian and replaces the time direction and the linear dilaton direction of (1.3); this should not be confused with the unrelated appearance of the Euclidean  $SL(2, \mathbb{R})/U(1)$  space in our discussions of the double scaling limit<sup>32</sup> (4.6). The behavior at finite temperatures may be analyzed by the Euclidean continuation of (9.1), which involves the same Euclidean  $SL(2)/U(1)$  whose correlation functions we discussed in detail above, although its interpretation now is quite different (since the direction around the cigar is now the Euclidean time direction). The canonical partition function of LST also receives contributions from the bulk geometry in which we just compactify the time direction of (1.3) on a circle,

---

<sup>31</sup> The discussion of this section applies only to LSTs with  $k > 1$ , since then the  $SL(2, \mathbb{R})/U(1)$  black hole corresponds to a normalizable state in the theory ( $k > 1$  is required so that in equations (2.48), (3.9),  $\beta = -1/Q < -Q/2$ ). In particular, among the examples we discussed above, it applies to the six dimensional example and to the four dimensional examples with  $n > 2$ , but not to two dimensional string theory, where the black hole is non-normalizable [27]. We thank J. Maldacena for a discussion on this issue.

<sup>32</sup> In the case of the six dimensional LSTs discussed in sections 4-6, the finite energy-density background (9.1) takes the form  $SL(2, \mathbb{R})_k/U(1) \times \mathbb{R}^5 \times SU(2)_k$ .

but (as in the AdS/CFT correspondence [58]) this is sub-dominant near the Hagedorn temperature.

In classical string theory, the radius of the (asymptotic) circle in the  $SL(2)/U(1)$  theory at level  $k$  is  $\sqrt{k\alpha'}$ . Thermodynamically, this corresponds to having  $\beta = 2\pi\sqrt{k\alpha'}$ , so the temperature is fixed at a value

$$T = T_H \equiv 1/2\pi\sqrt{k\alpha'}. \quad (9.2)$$

This temperature is independent of the energy density, which in (9.1) is a function of the string coupling  $g_s^{(tip)}$  at the horizon of the Lorentzian  $SL(2)/U(1)$  space,

$$\frac{E}{V} \simeq \frac{kM_s^d}{(g_s^{(tip)})^2}. \quad (9.3)$$

This is a Hagedorn-like behavior, corresponding in the micro-canonical ensemble to a density of states  $\rho(E) \simeq e^{E/T_H}$  (we will assume in this section that the  $(d-1)$ -dimensional space of the LST has been compactified on some finite large volume  $V$ , and we will not explicitly write the volume dependence).

Quantum corrections can change this behavior; they were analyzed at one-loop in string theory in [53]. It was found there that the classical density of states is corrected to

$$\rho(E) \simeq E^\alpha e^{E/T_H} \quad (9.4)$$

with a negative value of  $\alpha$  (smaller than  $(-1)$ ). This leads to a temperature which is slightly above the Hagedorn temperature (at large energy densities), and to a negative specific heat, meaning that the canonical ensemble is ill-defined.

The behavior (9.4) implies that as one gradually increases the temperature of LST, the Hagedorn temperature is reached at a finite energy density (below the energy densities for which (9.4) is a good approximation). Due to the negative specific heat, higher temperatures cannot be accommodated in the background (9.1). However, the fact that the Hagedorn temperature is reached at a finite energy density suggests that perhaps at this temperature there could be a transition to a different phase, as one finds (for instance) in  $SU(N)$  gauge theories with large  $N$ . As usual, the thermodynamic instability described above is reflected in the canonical ensemble by a winding mode of the string around the thermal time direction which is (classically) massless [53], and the phase transition would correspond to a condensation of this mode (as in [59]). We would like to argue that such a

phase transition actually does not occur, and that the Hagedorn temperature (9.2) really is a maximal temperature for LST.

The simplest argument for this comes from the micro-canonical ensemble. In this ensemble the black hole states (9.1) give rise to a density of states  $\rho(E) \sim e^{E/T_H}$  (at finite volume). These states are not exactly stable (due to Hawking radiation), but their life-time increases as  $E$  becomes larger. So, it seems that one can trust this prediction for the density of states at asymptotically high energies (at least, it is a lower bound on the density of states – there could be other types of states that would increase it further). The existence of such a density of states implies that the theory should not make sense at temperatures above the Hagedorn temperature, since all thermodynamic quantities would diverge there.

In order to study the same question in the canonical ensemble, we need to analyze the dynamics of the (classically massless) winding mode mentioned above in the Euclidean background (9.1), to compute its effective potential and to see if it condenses or not. The vertex operator for this mode is [53]

$$V_W = e^{-\varphi - \bar{\varphi}} V_{j;m,\bar{m}}^{(sl,susy)} e^{ip_\mu x^\mu}, \quad (9.5)$$

with  $m = \bar{m} = k/2$ , and the massless mode corresponds to the residue of the pole in this operator at  $j = m - 1 = (k - 2)/2$  (which exists only for  $k > 1$ ). As mentioned above, this normalizable mode could become unstable at one-loop order and destabilize the background. In the string theory the condensation of the corresponding normalizable state (with zero momentum) would be described by adding to the worldsheet Lagrangian the deformation

$$G_{-1/2} \bar{G}_{-1/2} V_{j;m,\bar{m}}^{(susy,norm)} + c.c. \quad (9.6)$$

with  $m = \bar{m} = k/2, j = (k - 2)/2$ .

The winding operator  $V_W$  in the Euclidean  $SL(2)/U(1)$  theory is precisely the same as the operator that corresponds to  $\tilde{\text{tr}}(B^k)$  in the six dimensional  $\mathcal{N} = (1, 1)$  DSLST, discussed in the previous sections (see equations (4.15), (4.19)). In the DSLST we saw that condensing this mode is equivalent to changing the string coupling at the tip of the



$SL(2, \mathbb{R})/U(1)$  cigar<sup>33</sup>. Thus, the same must be true in the background (9.1), where this is interpreted as changing the energy density. This is not surprising – in the thermodynamic context one may expect the instability of the thermal LST to be towards lowering or raising the energy of the system. This means that the tachyon condensation here would either increase the energy density, perhaps driving it to infinity (and the string coupling at the tip to zero), or decrease the energy density (where for small enough energy densities the perturbative analysis of (9.1) would break down).

It is important to note that the full spectrum of the string theory involving the Euclidean  $SL(2)/U(1)$  is quite different in the two cases (9.1) and (1.4), with different GSO projections, orbifolds and so on – but the particular operator  $V_W$  (9.5) appears in both cases. In particular, this operator has the same tree-level correlation functions in DSLST and in (9.1). This means that we can compute the tree-level potential for the would-be thermal tachyon  $V_W$  by computing zero-momentum correlation functions of the normalizable mode of  $\widetilde{\text{tr}}(B^k)$  in the six dimensional DSLST.

However, these all vanish, since we know that there is no potential for  $B$  in this (maximally supersymmetric) theory, so the S-matrix must vanish at zero momentum. This means that the potential for the would-be tachyon vanishes at tree-level. This again should not be surprising given the interpretation of this tachyon as the energy density, since the tree-level partition function of (9.1) vanishes at all energies.

Thus, the potential for the “winding tachyon”  $V_W$  first arises at the one-loop level. Moreover, since we interpreted this mode as the energy density, we already know what this potential is – it can simply be read off from the one-loop partition function of [53]. This gives a potential of the form  $(\alpha + 1)\log(E)$ . This form of the potential means that the massless mode  $V_W$  does not just become tachyonic at one-loop, but actually develops a tadpole, which (since  $\alpha < -1$ ) drives it towards large values of the energy. There is no stable end-point to this tadpole condensation process (the corrections to the one-loop contribution become smaller and smaller as the energy increases). This is consistent with

---

<sup>33</sup> Note that this is true even though naively the vertex operator (9.6) is not the same as the vertex operator for a change in the string coupling at the tip of the “throat,” which does not carry any winding number. The two different non-normalizable vertex operators create the same normalizable state, as can be seen by applying equation (2.62) to the supersymmetric case, so adding (9.6) to the worldsheet action is the same as adding the deformation changing the string coupling at the tip of the cigar. This is related to the (worldsheet supersymmetric version of the) FZZ duality [26,27] between  $SL(2)/U(1)$  and sine-Liouville theory.

the thermodynamics developed in [53] – trying to go above the Hagedorn temperature leads to a configuration (9.1) which has negative specific heat, so one is driven towards higher energy densities (where the temperature approaches the Hagedorn temperature). There seems to be no stable configuration at any temperature above the Hagedorn temperature, consistently with the discussion of the micro-canonical ensemble above.

To summarize, we described a self-consistent picture of the thermodynamics of LSTs with  $k > 1$ . In the micro-canonical ensemble the density of states is given by (9.4), which implies that the canonical ensemble is only well-defined below the Hagedorn temperature. As one goes up to this temperature one approaches a finite average energy density [53], but it is not possible to achieve thermal equilibrium at any higher temperature. The fact that the Hagedorn temperature is a maximal temperature is similar to the behavior in free string theory in flat space. In order to reach this conclusion we did not need to use any of the detailed results of the previous sections, but our arguments are based on the understanding (described in sections 2 and 3) of the normalizable states in LST backgrounds and their correlation functions.

## 10. Further comments on the results

### 10.1. General structure of correlation functions

Our discussion of perturbative string theory in backgrounds of the general form (1.4) leads to the following qualitative picture. The theory has two distinct (but coupled) sectors, one associated with the vicinity of the tip of the cigar, the other with the asymptotic region far from the tip. This is reflected in the spectrum of normalizable states, as well as in the structure of off-shell Green functions of the physical observables of the theory.

General observables carry both momentum and winding around the cigar, and depending on the amount of momentum and winding they are sensitive to different aspects of the physics. Consider, for example, pure winding modes on the cigar. Since the energy of a wound fundamental string decreases as it moves towards the tip of the cigar, wound strings experience an attractive potential towards the tip. It was shown in [23] that this potential has bound states, which are nothing but the principal discrete series states with  $m = \bar{m} = \pm(j + n)$ ,  $n = 1, 2, \dots$ . As we have seen in §2, the structure of the correlation functions of non-normalizable vertex operators (or off-shell Green functions) corresponding to wound strings reflects the presence of these bound states, via the appearance of LSZ poles as a function of the external momenta. Near such poles, the Green function is

dominated by the contribution of the bound states and thus is sensitive only to dynamics near the tip of the cigar.

An example of the above discussion in the background associated with  $k$  type IIB  $NS5$ -branes (4.7) is the operators  $\tilde{\text{tr}}(B^n)$  (4.16), (4.19). They correspond to pure winding modes on the cigar (in general with fractional winding), and create massless and massive principal discrete series states when acting on the vacuum.

The situation with pure momentum observables is quite different. Since the radius of the circle decreases as we move towards the tip of the cigar, the potential felt by these modes is repulsive [23], and they do not form bound states near the tip. The physical process encoded in the off-shell Green functions of these observables is scattering off the repulsive potential provided by the tip. In particular, the singularity structure of the Green functions of momentum operators is different from that of the winding modes. They do not exhibit LSZ poles associated with the principal discrete series states, and their singularities are instead due to bulk amplitudes in the cigar geometry<sup>34</sup>. An example of such operators in the fivebrane background (4.7) is the vertex operators  $\tilde{\text{tr}}(A^n)$  (4.20), whose correlation functions do not have LSZ poles, as we saw.

The momentum-carrying operators are not completely blind to the physics associated with the principal discrete series states. For example, one expects the operators  $\tilde{\text{tr}}(A^n)$  in the  $d = 6$  example to be able to create states with  $n$   $A$ -particles in the low-energy theory. As we saw, such processes do not contribute to tree-level correlation functions in the background (4.7), but they should contribute to higher-genus correlation functions.

For general observables, which carry both momentum and winding, there is a competition between the two effects, the repulsive potential due to the momentum, and the attractive one due to the winding. If the operator is “winding dominated,” *i.e.* if  $m, \bar{m}$  (2.5) have the same sign, we saw (2.22), (2.31) that it couples to bound states living near the tip of the cigar. Otherwise, it behaves like a momentum mode. Note that this is consistent with what one expects: the potential

$$V(R) = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 \quad (10.1)$$

is attractive (*i.e.*  $V'(R) > 0$ ) when  $|w|R/\alpha' > |n|/R$ , and is repulsive otherwise.

---

<sup>34</sup> As mentioned in §2, the amplitudes also have more conventional multiparticle singularities, that are familiar from other string theory backgrounds.

The above general picture helps to compare the analytic structure of the Green functions of string theory in (for example) the background (4.7) to expectations based on the low energy gauge theory of  $k$  type IIB  $NS5$ -branes. The massless gauge theory states correspond in the geometry (4.7) to the lowest lying principal discrete series states on the cigar. Thus, roughly speaking, the low energy gauge theory lives near the tip of the cigar. As explained above, the physics of winding modes near the LSZ poles corresponding to these massless states is indeed dominated by the vicinity of the tip. Thus, the low energy behavior of correlation functions of operators such as  $\widetilde{\text{tr}}(B^n)$  (4.19) is dominated by the gauge theory contribution<sup>35</sup>. On the other hand, amplitudes that involve momentum modes such as  $\widetilde{\text{tr}}(A^n)$  (4.20) do not have this property. Their low energy behavior is governed by the large  $\phi$  region, and therefore has a non-gauge theoretic origin.

Thus, we see that the low energy amplitudes of string theory on the background (4.7) are not entirely due to the contribution of the broken  $SU(k)$  gauge theory one normally associates with the fivebranes. They receive another contribution from a different source, which in the cigar description corresponds to the contribution of the region  $\phi \rightarrow \infty$ .

### 10.2. *Weak-weak coupling duality?*

The above discussion helps to resolve another puzzle raised by the construction of the double scaling limit (4.6). The equivalence between string theory in the background (4.7) and the fivebrane theory at a point along its Coulomb branch is expected to be an example of a gauge-gravity duality. Such dualities usually have the property that the two dual descriptions of the physics are never simple at the same time. For example, for coincident fivebranes, at low energies the gauge theory is expected to be weakly coupled six dimensional  $SU(k)$  gauge theory, while the bulk description, corresponding to the CHS geometry (4.1) is strongly coupled as  $\phi \rightarrow -\infty$ , and thus is not useful.

From this point of view, the low energy limit of the fivebrane theory in the double scaling limit (4.6), (4.9) is very puzzling. On the gauge theory side,  $k$  separated fivebranes at a point along the Coulomb branch are expected to be described in the IR by a  $U(1)^{k-1}$  gauge theory with sixteen supercharges, which of course is free in the IR. At the same time, the gravity description in terms of string propagation in the background (4.7) is also weakly coupled in the limit (4.9) and, if one wishes, it is possible to make the  $\alpha'$  corrections

---

<sup>35</sup> Except for the two-point function, which receives also bulk contributions, as discussed in §5.1.

small as well by sending  $k \rightarrow \infty$ . Thus, naively we seem to conclude that DSLST at low energies is an example of weak-weak coupling gauge-gravity duality.

The resolution is that, as we saw, the  $U(1)^{k-1}$  gauge theory gives only part of the contributions to the DSLST correlation functions at low energies, and there are additional contributions coming from the dynamics in the bulk of the cigar. Thus, while the gravity description (4.7) is indeed weakly coupled, there is no alternative weakly coupled description which captures all of the low-energy off-shell Green functions. In fact, we expect that if there is a gauge theory dual of the full LST in the double scaling limit, it is strongly coupled, even at low energies.

It is interesting that the gauge and gravity descriptions share a weakly coupled sector that *can* be described in two different ways – the broken  $SU(k)$  field theory which can be described both by field theory methods and by focusing on the residues of the massless LSZ poles in string theory on (4.7). This is similar to the fact that certain correlation functions of chiral operators in  $\mathcal{N} = 4$  SYM can be computed either by studying the dynamics of gravitons on  $AdS_5$ , or by calculations in weakly coupled gauge theory. Presumably, the dynamics of this weakly coupled sector of LST is similarly constrained.

### 10.3. The density of normalizable states at large energies

In sections 2 and 3 we saw that normalizable states in DSLST correspond to principal discrete series states on the cigar. We would like to estimate the growth in the number of such states at large masses.

A rough way to proceed is as follows. A given operator with “more winding than momentum” (*i.e.* the same sign of  $m$  and  $\bar{m}$  (3.5)) can create a number of states that is proportional to  $(k|w| - |n|)$ . This grows linearly with the winding, and therefore with the mass of the state. However, a much larger contribution to the growth in the number of normalizable states at large mass comes from the exponential growth in the number of different operators that can create normalizable states<sup>36</sup> in (1.4). At high excitation levels, the growth in the number of such operators is comparable to that of ten dimensional string theory. It is somewhat smaller, since it is determined by the effective central charge of the worldsheet CFT, which is smaller than its actual (critical) central charge because it partly comes from an asymptotically linear dilaton background.

---

<sup>36</sup> In general one needs to be careful because the same state can be created by different operators, but we do not expect this to drastically change the arguments below.

Thus, we conclude that the high energy growth in the density of normalizable states in DSLST is exponential,  $\rho_{\text{pert}}(E) \sim \exp(\beta_{\text{pert}} E)$ , with  $\beta_{\text{pert}}$  of order one in string units. Of course, as implied by the notation, this only takes into account perturbative string states in the background (1.4). This estimate is expected to be reliable for energies well above  $M_s$  but well below  $M_W$  (or more generally well below  $M_s/g_s^{(\text{tip})}$ ). Around the scale  $M_W$ , non-perturbative states such as the W-bosons in the six dimensional example start appearing, and the perturbative estimate of the density of states breaks down. For energies well above  $M_W$  the density of states of LST is similar to that of a free superstring theory in  $4k + 2$  dimensions [49,60,50]. It is much larger than the density of perturbative states for all  $k \geq 2$ . Clearly, most of the high energy states of LST are non-perturbative, even when we are at the DSLST point in the moduli space.

#### 10.4. High-energy scattering in DSLST

As discussed in §1, §2, the S-matrix of normalizable states in DSLST is obtained by studying correlation functions of the corresponding normalizable vertex operators. It is interesting to examine the high energy behavior of this S-matrix (*i.e.* its behavior for large values of the Mandelstam invariants).

In fact, this analysis is identical to that in critical string theory. As mentioned in §1, the normalizable vertex operators on spacetimes like (4.7) behave essentially as those in the critical string. Thus, the momentum dependence of the S-matrix is precisely equal to that of the usual superstring in  $\mathbb{R}^{d-1,1}$  (recall that the  $SL(2, \mathbb{R})/U(1)$  component of normalizable vertex operators does not depend on the momentum  $p_\mu$  in  $\mathbb{R}^{d-1,1}$ ). This means that the high-energy behavior of the S-matrix in DSLST will be similar to that of standard string theories. It will exhibit Regge behavior in the appropriate regime, and will decay exponentially when all kinematic variables are large. This is another aspect of LSTs which is similar to that of standard string theories, even though they are not gravitational.

In standard string theories the behavior described above can be trusted up to energies of the order of the Planck scale, where higher orders in string perturbation theory and non-perturbative effects (such as black holes) become important. Similarly, we expect that in DSLST this behavior will persist until the scale  $M_W$ , but will be modified above this scale.

## 11. Open problems

The results of this paper lead to a number of questions. In this section we would like to briefly discuss some of them.

### 11.1. *Little string worldsheets*

In studying the analytic structure of off-shell Green functions in spacetimes of the general form (1.4), we found it convenient to express the  $SL(2, \mathbb{R})/U(1)$  vertex operators in terms of vertex operators on  $AdS_3$ ,  $\Phi_j(x, \bar{x})$ , integrated over the variables  $(x, \bar{x})$  (see *e.g.* (2.17)). The correlation functions on the cigar can then be written as integrated versions of the  $AdS_3$  ones (2.36).

Even though  $x$  itself is not a meaningful object in string theory on the cigar (in contrast to the  $AdS_3$  case), we saw that the analytic structure of the correlation functions on the cigar is usefully described by studying various degeneration limits of the integrals over the  $x_i$ . For example, the contributions of the regions  $x_i \rightarrow 0, \infty$  (two points that are picked arbitrarily at the outset by the definition of the integral transform (2.17)) were seen to give rise to LSZ poles associated with external legs going on-shell, while contributions from regions where two or more of the  $x_i$  approach each other were shown to give rise to singularities associated with bulk interactions.

It is natural to ask whether the variables  $(x, \bar{x})$  are just a convenient technical tool for analyzing the analytic structure of the amplitudes, or whether they have a deeper physical significance. Recall that in string theory on  $AdS_3$  these variables label positions on the boundary of spacetime, and thus describe the base space on which the two dimensional *spacetime* CFT is living. It is natural to conjecture that in LST they should be thought of as worldsheet variables for some sort of strings, in terms of which the dynamics can be formulated.

Indeed, the singularities of amplitudes that we find arise in precisely the right way for such an interpretation. We can interpret  $x = 0$  as corresponding to the far past on the worldsheet (in the sense of radial quantization, as is standard in CFT, or by mapping the  $x$  plane to a cylinder), and  $x = \infty$  as the far future. Thus, singularities associated with these regions have to do with the contribution of on-shell physical states, as we have found. Similarly, the fact that regions in which some of the  $x_i$  approach each other have to do with interactions is familiar from studies of Shapiro-Virasoro amplitudes in critical string theory.

It would be interesting to reformulate our results in terms of the dynamics of the strings whose worldsheet is labeled by  $(x, \bar{x})$ . It is clear that this would be a very different kind of string theory from what we are accustomed to, and the worldsheet description is bound to be different as well. For example, this string theory is non-critical, off-shell amplitudes in spacetime make sense in it, and there does not seem to be an analog of the volume of the conformal Killing group that we divide by, that in the usual string theory fixes three of the  $n$  integrals in an  $n$  point function. It is also not clear whether/how one is supposed to sum over the genus of the worldsheet labeled by  $x$ , and if so, with what weight (*i.e.* what is the value of the string coupling), etc. As described above, the  $x$  coordinates seem to naturally live on a sphere with punctures.

It is interesting to note that some type of “little string” appears also in the DLCQ description of LSTs, at least for the case of the  $d = 6$   $\mathcal{N} = (1, 1)$  LSTs which we discussed in detail above. The DLCQ of these theories, with  $N$  units of light-like momentum, is given by the  $1 + 1$  dimensional SCFT which arises at low energies on the Coulomb branch of the  $\mathcal{N} = (4, 4)$   $U(N)^k$  gauge theory with bifundamental hypermultiplets [61,62,8], compactified on a circle. The Coulomb branch includes configurations where each  $U(N)$  group is broken to an Abelian subgroup, and as in the Matrix theory description of type IIA string theory [63,64,65] one can construct “long string” configurations in which the eigenvalues of the matrices are permuted around the circle, and which could carry energies of order  $1/N$ . It is tempting to conjecture that these strings could be related to the strings mentioned in the previous paragraphs. However, there is no reason to believe that these strings are weakly coupled, so it is hard to see why correlation functions on their worldsheet would be meaningful.

### *11.2. Other asymptotically linear dilaton spacetimes*

Throughout most of this paper we focused on backgrounds of the form (1.4), which contain an  $SL(2, \mathbb{R})/U(1)$  factor, and used the fact that in that case we know a lot about the CFT. In particular, the variables  $(x, \bar{x})$ , which we have used extensively, appear due to the fact that the worldsheet CFT is a coset of  $SL(2, \mathbb{R})$ . It is natural to wonder how the structure of the theory changes when we consider more general asymptotically linear dilaton spacetimes.

For example, in the six dimensional LST with  $\mathcal{N} = (1, 1)$  supersymmetry that we focused on in this paper, such backgrounds can be obtained by moving away from the highly symmetric point in moduli space (4.4) to more generic points, corresponding to other



distributions of  $NS5$ -branes in the transverse  $\mathbb{R}^4$ . For all points in the moduli space, the background looks the same near the boundary (4.1); what distinguishes between different points in the moduli space is the form of the “wall” that prevents  $\phi$  from going to  $-\infty$ . The cigar, or  $\mathcal{N} = 2$  Liouville (3.9), wall is replaced by a more general one.

Physically, one would expect most of the qualitative conclusions we reached to be valid in this more general setup. There should still be normalizable states living in the vicinity of the wall. These should include the massless gauge bosons living on the fivebranes, which should of course be there anywhere in moduli space. Some of the correlation functions of non-normalizable (off-shell) operators should exhibit LSZ poles associated with these states. There should also be another sector of the theory that is sensitive to the structure far from the wall, and amplitudes that involve the relevant operators should exhibit bulk poles. The type of UV/IR mixing that we found should also exist more generally than our examples.

It seems that in order to verify the above claims, one has to develop more powerful techniques for studying string theory in asymptotically linear dilaton spacetimes, and in particular the analytic structure of amplitudes in such spacetimes. Of special interest is the question whether there is an analog of the variables  $(x, \bar{x})$  that allows one to analyze the analytic structure of amplitudes in a way similar to what we have done in the case of  $SL(2, \mathbb{R})/U(1)$  above. If the interpretation of  $(x, \bar{x})$  in terms of worldsheet variables for little strings is correct, it seems that such variables should exist at any point in moduli space.

### *11.3. Phenomenological implications*

One of the conclusions of our analysis was that some correlation functions in LST, such as the two-point function of §5.2, are enhanced in the IR (due to the bulk poles) in a way that cannot be understood just by studying the light degrees of freedom. This effect violates the usual renormalization group intuition in which low-energy correlators can be described just by using low-energy states. It is natural to ask whether such effects could possibly have phenomenological implications in compactifications of string theory that include LSTs, such as the scenario of [66], where the asymptotically linear dilaton direction is cut off by an upper bound on  $\phi$ , but the value of the string coupling there (which determines the four dimensional Planck scale) is very small.

Observing the resulting IR enhancement seems to be difficult for the following reason. This enhancement occurs in correlation functions of operators whose wave-function is dominated by the weak-coupling end of the “throat.” However, in these scenarios, the standard model fields do not live at the (very) weakly coupled end of the “throat,” but rather near the wall (for instance, they could include the principal discrete series states living near the tip of the cigar in the example of this paper, or D-branes living near the tip). It is not clear how observers made of standard model fields can access the analytic structure encoded in the correlation functions of operators whose wave-function is dominated by the other end of the “throat”<sup>37</sup>. On the other hand, the fact (discussed in §9) that LSTs have a limiting temperature at the string scale could potentially have observable cosmological consequences in the scenario of [66], where  $M_s$  is around a TeV.

#### 11.4. Open-closed string duality

It is natural to ask whether asymptotically linear dilaton spacetimes such as (1.3), (1.4) have a holographically dual open string description. In the special case of two dimensional string theory discussed in section 8, there is a well known dual – matrix quantum mechanics in the double scaling limit [45,17,46]. This is now understood as the theory on a large number of D-branes localized deep inside the Liouville wall [67,68]. A generalization of this construction leads to a matrix model dual to two dimensional type 0 string theory [69,70]. Some results on matrix models dual to two dimensional string theory on the cigar are available as well [27,71,72].

This naturally leads to the question of whether one can similarly construct a matrix model dual to LST backgrounds (1.4) in higher dimensions. One way to proceed is to consider again the dynamics on D-branes localized in the region where the coupling is largest, which on the cigar is the region near the tip. A number of questions immediately arise. One is how many such branes should we take. In the AdS/CFT correspondence, the number of  $D3$ -branes is in general finite and related to the string coupling on  $AdS_5$ , while in the holographic duality of two dimensional string theory mentioned above, the matrix model has strictly infinite  $N$  (in the double scaling limit).

Another question is in how many dimensions of  $\mathbb{R}^{d-1,1}$  should the branes be extended. Naively, one might expect them to be extended in all  $d$  dimensions, so as to match the Poincaré symmetry of the closed string dual, but it is possible that other choices are allowed

---

<sup>37</sup> We thank S. Dimopoulos and E. Silverstein for discussions on this issue.

as well. Finally, one can ask whether the open string dual is expected to be the full open string theory on these branes, or just some low energy limit of the full open string theory.

The answers to all these questions are not clear at present. We believe that it is likely that one should consider an infinite number of localized D-branes, but obviously more work is required to definitively decide one way or the other. In any case, it would be very interesting to rederive some or all of our results from an open string perspective.

It might also be possible to explore the physics of LST by using its “deconstruction” by a large  $N$  gauge theory, either along the lines of [73] or by using the new deconstruction of LST in [74]. In this framework the large  $N$  gauge theory is also responsible for the construction of (typically two – either compact or large) dimensions in LST.

In [74], a certain deformation of four dimensional  $\mathcal{N} = 4$  SYM with gauge group  $SU(N)$  at large  $N$  was considered. This theory has a branch where it is confined down to  $SU(k)$  (or  $U(1)^{k-1}$  in the Coulomb branch of the unconfined sector). In a certain scaling limit on this branch at large  $N$  it is claimed that this gauge theory theory is equivalent to six dimensional DSLST. It would be very interesting to make the correspondence more precise. In particular, the construction of [74] raises the interesting possibility that some of the peculiar low energy results that we found using the bulk description reflect the physics associated with a strongly coupled sector of the dual gauge theory.

**Acknowledgements:** We would like to thank Y. Antebi, M. Berkooz, S. Dimopoulos, N. Dorey, B. Fiol, E. Kiritsis, J. Maldacena, A. Naqvi, V. Niarchos, N. Seiberg, E. Silverstein and T. Volansky for useful discussions. O.A. would like to thank Stanford University, SLAC and Harvard University for hospitality during the course of this project. A.G. would like to thank the EFI and the Department of Physics at the University of Chicago, where this work was initiated, for its warm hospitality. D.K. thanks the Weizmann Institute of Science, LPTHE Paris VI and the NHETC at Rutgers University for hospitality. The work of O.A. was supported in part by the Israel-U.S. Binational Science Foundation and by Minerva. O.A. is the incumbent of the Joseph and Celia Reskin career development chair. The work of A.G. was supported in part by the German-Israel Bi-National Science Foundation. The work of O.A. and A.G. is supported in part by the Israel Academy of Sciences and Humanities – Centers of Excellence Program, by the European network HPRN-CT-2000-00122, and by the Albert Einstein Minerva Center for Theoretical Physics. Minerva is funded through the BMBF. D.K. is supported in part by DOE grant #DE-FG02-90ER40560.

## Appendix A. Some results from $SL(2)/U(1)$ CFT

### A.1. Review of two-point and three-point functions in the $SL(2)$ and $SL(2)/U(1)$ CFTs

We start our discussion from the bosonic  $SL(2)$  WZW model of level

$$k_{SL(2)} = k + 2, \quad (\text{A.1})$$

with central charge

$$c_{SL(2)} = \frac{3(k+2)}{k}. \quad (\text{A.2})$$

The natural observables in the theory defined on the Euclidean version of  $SL(2)$ ,  $H_3^+ \equiv SL(2, \mathbb{C})/SU(2)$ , are primaries  $\Phi_j(x, \bar{x})$  of the  $SL(2)_L \times SL(2)_R$  current algebra [18,19] with  $j > -\frac{1}{2}$ . The worldsheet scaling dimension of  $\Phi_j(x, \bar{x})$  is

$$\Delta(j) = -\frac{j(j+1)}{k}. \quad (\text{A.3})$$

In the papers [18,19] the operators  $\Phi_j$  are normalized as follows:

$$\langle \Phi_{j_1}(x_1, \bar{x}_1) \Phi_{j_2}(x_2, \bar{x}_2) \rangle = \delta(j_1 - j_2) \frac{k}{\pi} \left[ \frac{1}{k\pi} \gamma \left( \frac{1}{k} \right) \right]^{2j_1+1} \gamma \left( 1 - \frac{2j_1+1}{k} \right) |x_{12}|^{-4(j_1+1)}, \quad (\text{A.4})$$

with  $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$ . In some of our computations (in particular in §5.4) it is more convenient to choose a different normalization

$$\tilde{\Phi}_j(x, \bar{x}) \equiv \frac{\Phi_j(x, \bar{x})}{\sqrt{\frac{k}{\pi} \left[ \frac{1}{k\pi} \gamma \left( \frac{1}{k} \right) \right]^{2j+1} \gamma \left( 1 - \frac{2j+1}{k} \right)}}. \quad (\text{A.5})$$

In this normalization the two-point function is

$$\langle \tilde{\Phi}_{j_1}(x_1, \bar{x}_1) \tilde{\Phi}_{j_2}(x_2, \bar{x}_2) \rangle = \delta(j_1 - j_2) |x_{12}|^{-4(j_1+1)}. \quad (\text{A.6})$$

Most of our considerations in this paper are independent of the normalization, since it does not affect the pole structure for  $-1/2 < j < (k-1)/2$ ; it is easy to translate the formulas below to the normalization (A.5).

For discussing the coset  $SL(2)/U(1)$  it is convenient to choose a different basis for the primaries  $\Phi_j$  (or  $\tilde{\Phi}_j$ )

$$\Phi_{j;m,\bar{m}} = \int d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_j(x, \bar{x}). \quad (\text{A.7})$$

The two-point function in this basis was computed in [26,14] :

$$\langle \Phi_{j;m,\bar{m}} \Phi_{j';-m,-\bar{m}} \rangle = k \left[ \frac{1}{k\pi} \gamma \left( \frac{1}{k} \right) \right]^{2j+1} \gamma \left( 1 - \frac{2j+1}{k} \right) \delta(j-j') \times \frac{\Gamma(-2j-1)\Gamma(j-m+1)\Gamma(1+j+\bar{m})}{\Gamma(2j+2)\Gamma(-j-m)\Gamma(\bar{m}-j)}. \quad (\text{A.8})$$

The three-point function in the  $SL(2)$  CFT takes the form

$$\langle \Phi_{j_1}(x_1, \bar{x}_1) \Phi_{j_2}(x_2, \bar{x}_2) \Phi_{j_3}(x_3, \bar{x}_3) \rangle = D(j_1, j_2, j_3) |x_{12}|^{2(j_3-j_1-j_2-1)} |x_{13}|^{2(j_2-j_1-j_3-1)} |x_{23}|^{2(j_1-j_2-j_3-1)}, \quad (\text{A.9})$$

where the structure constants  $D(j_1, j_2, j_3)$  were computed in [18,19]:

$$D(j_1, j_2, j_3) = \frac{k}{2\pi^3} \left[ \frac{1}{k\pi} \gamma \left( \frac{1}{k} \right) \right]^{j_1+j_2+j_3+1} \times \frac{G(-j_1-j_2-j_3-2)G(j_3-j_1-j_2-1)G(j_2-j_1-j_3-1)G(j_1-j_2-j_3-1)}{G(-1)G(-2j_1-1)G(-2j_2-1)G(-2j_3-1)}. \quad (\text{A.10})$$

$G(j)$  is a special function which satisfies the following useful identities:

$$\begin{aligned} G(j) &= G(-j-1-k), \\ G(j-1) &= \gamma(1 + \frac{j}{k}) G(j), \\ G(j-k) &= k^{-(2j+1)} \gamma(j+1) G(j). \end{aligned} \quad (\text{A.11})$$

$G(j)$  has poles at the following values of  $j$ :  $j = n + mk$ ,  $j = -(n+1) - (m+1)k$ , where  $n, m = 0, 1, 2, \dots$ . In particular, for  $j = 0, 1, 2, \dots < k$  it has single poles.

In the  $(j, m, \bar{m})$  basis the three-point function for the special case  $m = \bar{m}$  is given by

$$\langle \Phi_{j_1;m_1,m_1} \Phi_{j_2;m_2,m_2} \Phi_{j_3;m_3,m_3} \rangle = D(j_1, j_2, j_3) \times F(j_1, m_1; j_2, m_2; j_3, m_3) \int d^2x |x|^{2(m_1+m_2+m_3-1)}, \quad (\text{A.12})$$

where

$$F(j_1, m_1; j_2, m_2; j_3, m_3) = \int d^2x_1 d^2x_2 |x_1|^{2(j_1+m_1)} |x_2|^{2(j_2+m_2)} \times |1-x_1|^{2(j_2-j_1-j_3-1)} |1-x_2|^{2(j_1-j_2-j_3-1)} |x_1-x_2|^{2(j_3-j_1-j_2-1)}. \quad (\text{A.13})$$

The integral over  $x$  in (A.12) ensures momentum conservation  $m_1 + m_2 + m_3 = 0$ . The function  $F$  (A.13) does not seem to be expressible in terms of elementary functions.

The same two-point functions and three-point functions arise also in the coset  $SL(2)/U(1)$  for the operators  $V_{j;m,\bar{m}}$  (arising from  $\Phi_{j;m,\bar{m}}$ ) when we look at correlation functions preserving the winding number, since the  $U(1)$  part contributes trivially. In the coset, additional correlation functions are non-vanishing as well.

### A.2. The superconformal algebra of $SL(2)/U(1)$

The supersymmetric  $SL(2)$  WZW model of level  $k$  may be viewed as the sum of a bosonic  $SL(2)$  theory of level  $k+2$ , with currents  $j^a$  ( $a = 1, 2, 3$ ),

$$j^a(z)j^b(0) \sim \frac{\frac{1}{2}(k+2)\eta^{ab}}{z^2} + i\epsilon^{abc}\frac{j_c(0)}{z}, \quad (\text{A.14})$$

( $\eta = \text{diag}(1, 1, -1)$ ) and of three free fermions  $\lambda^a$  ( $a = 1, 2, 3$ ), which can be associated with an  $SL(2)$  current algebra of level  $k = -2$ . The total  $SL(2)$  currents are given by

$$J_a^{(\text{total})} = j_a - \frac{i}{2}\epsilon_{abc}\lambda^b\lambda^c, \quad (\text{A.15})$$

where the fermions obey

$$\lambda^a(z)j^b(w) \sim 0, \quad \lambda^a(z)\lambda^b(w) \sim \frac{\eta^{ab}}{z-w}. \quad (\text{A.16})$$

This theory has an  $\mathcal{N} = 1$  superconformal symmetry generated by the current

$$G = Q(\eta_{ab}\lambda^a j^b - \frac{i}{6}\epsilon_{abc}\lambda^a\lambda^b\lambda^c), \quad (\text{A.17})$$

with  $Q^2 = 2/k$ . The  $\lambda^a$ 's are superconformal primaries, and the  $J_a^{(\text{total})}$  are the top components of the corresponding multiplets.

The supersymmetric  $SL(2)/U(1)$  theory is defined by gauging the  $U(1)$  superfield including  $\lambda^3$  and  $J_3^{(\text{total})}$ . This theory has an  $\mathcal{N} = 2$  superconformal algebra, generated by

$$\begin{aligned} G^+ &= QJ^-\lambda^+, & G^- &= QJ^+\lambda^-, \\ J &= (1 + Q^2) : \lambda^+\lambda^- : + Q^2 j^3 = : \lambda^+\lambda^- : + Q^2 J^{3(\text{total})}, \end{aligned} \quad (\text{A.18})$$

where

$$J^\pm \equiv J^1 \pm iJ^2, \quad \lambda^\pm \equiv \frac{1}{\sqrt{2}}(\lambda^1 \pm i\lambda^2). \quad (\text{A.19})$$

The right-moving fields obey similar algebras.

## Appendix B. Mixing of single and multi-trace operators

As we saw in the text, the single string vertex operators in the background (4.1) do not correspond to single trace operators in the low energy  $SU(k)$  gauge theory, but rather to a

mixture of single and multi trace operators. In this appendix we determine the coefficients of the different multi trace operators in this mixture.

The operators of interest to us will be

$$\mathcal{O}_n \equiv \frac{1}{n} \text{tr}(B^n). \quad (\text{B.1})$$

The correlation functions of interest are

$$(n_1, n_2, \dots, n_j) \equiv \langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \cdots \mathcal{O}_{n_j} \bar{\mathcal{O}}_n \rangle; \quad n = \sum_{i=1}^j n_i. \quad (\text{B.2})$$

As usual in DSLST, we will compute these correlation functions at a point along the Coulomb branch, (4.4), where the  $SU(k)$  gauge symmetry is broken to  $U(1)^{k-1}$ . Using free field theory, one finds that the correlators (B.2) are non-zero. The leading connected diagram in the  $1/M_W$  expansion is due to a contraction of a single  $B$  out of each  $\mathcal{O}_{n_i}$  with a  $B^*$  inside the  $\bar{\mathcal{O}}_n$ . Thus, it goes like  $M_W^{2(n-j)}$ , and the dependence on the locations of the operators,  $(x_1, x_2, \dots, x_j; \bar{x})$ , is simple,  $(n_1, n_2, \dots, n_j) \sim \prod_{i=1}^j (x_i - \bar{x})^{-4}$  (we will omit it below). Our main interest will be on the dependence on the  $\{n_i\}$ ; one finds<sup>38</sup>

$$(n_1, n_2, \dots, n_j) = k(n-1)(n-2) \cdots (n-j+1) = k \frac{\Gamma(n)}{\Gamma(n-j+1)} \quad (\text{B.3})$$

where  $n = \sum_i n_i$  and  $k$  is the number of fivebranes (or the rank of the gauge group plus one).

The string theory analysis of §6 shows that the single string vertex operators with the quantum numbers of  $\mathcal{O}_n$ , which were denoted by  $\mathcal{B}_n$  in §6, have the property that the analogs of the correlators (B.2), (B.3) vanish for them (to leading order in the  $1/M_W$  expansion). Our interpretation of this fact is that the correspondence between  $\mathcal{O}_n$  and  $\mathcal{B}_n$  is non-trivial. The symmetries allow a general mixing of the form

$$\mathcal{B}_n = \mathcal{O}_n + \sum_l \frac{1}{l!} \alpha_{n_1, n_2, \dots, n_l} \mathcal{O}_{n_1} \cdots \mathcal{O}_{n_l} \quad (\text{B.4})$$

where the integers  $n_i$  are summed over, subject to the constraint that their sum is  $n$ . The basic idea is that to match the vanishing of the string theory correlation function in the low energy gauge theory, one should consider instead of (B.2) the correlation function  $\langle \mathcal{B}_{n_1} \mathcal{B}_{n_2} \cdots \mathcal{B}_{n_j} \bar{\mathcal{B}}_n \rangle$  and fix the coefficients  $\alpha_{n_1, n_2, \dots, n_l}$  such that it vanishes. It is clear

---

<sup>38</sup> For  $j = 1$  one has  $(n) = k$ .

that there is the same number of free parameters ( $\alpha_{n_1, n_2, \dots, n_l}$ ) and equations (due to the vanishing of the above correlators). So, we expect to be able to find a solution.

As we will see, something surprising (from the current perspective) happens, and the coefficients  $\alpha$  that one finds this way actually do not depend on all the  $n_i$ , but only on their sum,  $n$ , and on  $l$ . Thus, we will denote the coefficients  $\alpha$  by  $\alpha_{n;l}$ . We will find that they are given by the simple expression

$$\alpha_{n;l} = \left( \frac{1-n}{k} \right)^{l-1}. \quad (\text{B.5})$$

The mixing (B.4) leads to the following expansion for the correlation function of interest:

$$\begin{aligned} 0 = \langle \mathcal{B}_{n_1} \mathcal{B}_{n_2} \cdots \mathcal{B}_{n_j} \bar{\mathcal{B}}_n \rangle = & (n_1, n_2, \dots, n_j) + \alpha_{n;2} \sum (n, \dots, n)(n, \dots, n) \\ & + \alpha_{n;3} \sum (n, \dots, n)(n, \dots, n)(n, \dots, n) + \cdots, \end{aligned} \quad (\text{B.6})$$

where in each term one sums over all different orderings of the  $n_i$ . We next give the few lowest of these equations, and use them to determine  $\alpha_{n;l}$  for small  $l$ .

For  $j = 2$ , we have

$$(n_1, n_2) + \alpha_{n;2}(n_1)(n_2) = 0 \quad (\text{B.7})$$

Using (B.3) to evaluate the first term, and the fact that  $(n_1) = (n_2) = k$ , we find that  $\alpha_{n;2} = -(n-1)/k$ , in agreement with the general expression (B.5).

For  $j = 3$ , we have

$$(n_1, n_2, n_3) + \alpha_{n;2} [(n_1, n_2)(n_3) + (n_1, n_3)(n_2) + (n_2, n_3)(n_1)] + \alpha_{n;3}(n_1)(n_2)(n_3) = 0 \quad (\text{B.8})$$

Using (B.3) and the form of  $\alpha_{n;2}$  found previously, one concludes that

$$k(n-1)(n-2) - k^2 \frac{n-1}{k} (2n-3) + k^3 \alpha_{n;3} = 0 \quad (\text{B.9})$$

This leads to  $\alpha_{n;3} = (n-1)^2/k^2$ , again in agreement with (B.5).

For  $j = 4$ , we have (in hopefully self explanatory notation)

$$\begin{aligned} & (1, 2, 3, 4) + \\ & \alpha_{n;2} [(1, 2)(3, 4) + (1, 3)(2, 4) + (1, 4)(2, 3) + \\ & (1, 2, 3)(4) + (1, 2, 4)(3) + (1, 3, 4)(2) + (2, 3, 4)(1)] + \\ & \alpha_{n;3} [(1, 2)(3)(4) + (1, 3)(2)(4) + (1, 4)(2)(3) + \\ & (2, 3)(1)(4) + (2, 4)(1)(3) + (3, 4)(1)(2)] + \\ & \alpha_{n;4}(1)(2)(3)(4) = 0. \end{aligned} \quad (\text{B.10})$$



Substituting the known results one finds

$$(n-1)(n-2)(n-3) - (n-1)(3n^2 - 12n + 11) + (n-1)^2(3n-6) + k^3\alpha_{n;4} = 0 \quad (\text{B.11})$$

Thus,  $\alpha_{n;4} = -(n-1)^3/k^3$ , as in (B.5).

For  $j = 5$ , the full expression is somewhat long, so we only give the analog of (B.11) for this case. It is:

$$\begin{aligned} & (n-1)(n-2)(n-3)(n-4) - (n-1)(4n^3 - 30n^2 + 70n - 50) + \\ & (n-1)^2(6n^2 - 30n + 35) - (n-1)^3(4n-10) + k^4\alpha_{n;5} = 0. \end{aligned} \quad (\text{B.12})$$

Again, this is in agreement with (B.5).

A direct continuation of the above approach to higher values of  $j$  seems impractical. To get more general results we will use a property of the calculations described above that seems quite non-trivial but that we have not been able to prove in general. Apriori, one might expect the coefficient of each  $\alpha_{n;l}$  in equations like (B.6), (B.8) – (B.12) to depend on the individual  $n_i$ , but the explicit formulae displayed above *always* depend only on  $n = \sum n_i$ . This is why the coefficients  $\alpha$  in (B.4) can be taken to depend only on  $n$  and  $l$ .

If we assume this property persists for arbitrarily high  $l$  (an assumption that is perhaps not unlikely since we have seen that all terms with  $l \leq 5$  do satisfy this property), we can choose particularly convenient values for the  $n_i$ , and use them to determine the mixing coefficients  $\alpha_{n;l}$ . This is what we do next.

Consider the special case

$$(n_1, \dots, n_j) = (n, 0, 0, \dots, 0) \quad (\text{B.13})$$

This is outside the range of interest for our application, but for proving the mathematical statement about polynomials it is as good as any other choice. In order to study this case we need to know what are  $(0, 0, \dots, 0) \equiv (0^l)$ , and  $(n, 0^j)$ . Using (B.3) it is not difficult to see that

$$\begin{aligned} (0^l) &= k(-1)^{l-1}(l-1)! \\ (n, 0^j) &= k \frac{(n-1)!}{(n-j-1)!} \end{aligned} \quad (\text{B.14})$$

The expansion (B.6) simplifies significantly in this case. One can write it as follows:

$$\frac{1}{j!}(n, 0^j) + \sum_{j_1+j_2=j} \frac{\alpha_{n;2}}{j_1!j_2!}(n, 0^{j_1})(0^{j_2}) + \sum_{j_1+j_2+j_3=j} \frac{\alpha_{n;3}}{2!} \frac{1}{j_1!j_2!j_3!}(n, 0^{j_1})(0^{j_2})(0^{j_3}) + \dots = 0. \quad (\text{B.15})$$

Now, define the following function of an auxiliary variable  $x$ :

$$f_n(x) = \sum_{j=0}^{\infty} \frac{1}{j!} (n, 0^j) x^j = k(1+x)^{n-1}. \quad (\text{B.16})$$

Note that although the sum over  $j$  runs all the way to infinity,  $f_n(x)$  is actually a polynomial, since the coefficients  $(n, 0^j)$  vanish for  $j \geq n$ . This is particularly clear in the representation with  $\Gamma$  functions in (B.3). In order to prove that the  $\alpha_{n;l}$  take the form (B.5) we would like to show that multiplying (B.15) by  $x^j$  and summing over  $j$  gives a vanishing answer (or more precisely an  $x$  independent constant – see below).

The sum over  $j$  actually simplifies (B.15) significantly, since now one can sum independently over  $j_1, j_2$ , etc. To be precise,  $j_1$  is now summed from 0 to infinity, while  $j_2, j_3$ , etc are summed from 1 to  $\infty$ . Performing the sum over the  $j_i$  (using (B.5)) one finds the following expression:

$$f_n(x) \sum_{l=1}^{\infty} \frac{(1-n)^{l-1}}{(l-1)!} [\log(1+x)]^{l-1} = f_n(x) e^{(1-n) \log(1+x)} = f_n(x) (1+x)^{1-n} = k. \quad (\text{B.17})$$

We see that we did not get quite zero, but the only non-zero term is a constant. It is easy to understand where it comes from. Equation (B.15) only holds for  $j > 0$ ; for  $j = 0$  only the first term is there, and it is equal to  $(n) = k$ . This is the non-zero term that we found in (B.17). We see that under the assumption that the  $\alpha$ 's really only depend on  $n$  and  $l$ , they must be given by the expression (B.5).

## References

- [1] O. Aharony, “A brief review of ‘little string theories’,” *Class. Quant. Grav.* **17**, 929 (2000) [arXiv:hep-th/9911147].
- [2] D. Kutasov, “Introduction to little string theory,” *Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001*.
- [3] O. Aharony, B. Fiol, D. Kutasov and D. A. Sahakyan, “Little string theory and heterotic/type II duality,” *Nucl. Phys. B* **679**, 3 (2004) [arXiv:hep-th/0310197].
- [4] M. Berkooz, M. Rozali and N. Seiberg, “On transverse fivebranes in M(atrix) theory on  $T^{*5}$ ,” *Phys. Lett. B* **408**, 105 (1997) [arXiv:hep-th/9704089].
- [5] N. Seiberg, “New theories in six dimensions and matrix description of M-theory on  $T^{*5}$  and  $T^{*5}/Z(2)$ ,” *Phys. Lett. B* **408**, 98 (1997) [arXiv:hep-th/9705221].
- [6] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, “Matrix description of interacting theories in six dimensions,” *Adv. Theor. Math. Phys.* **1**, 148 (1998) [arXiv:hep-th/9707079].
- [7] E. Witten, “On the conformal field theory of the Higgs branch,” *JHEP* **9707**, 003 (1997) [arXiv:hep-th/9707093].
- [8] O. Aharony and M. Berkooz, “IR dynamics of  $d = 2$ ,  $\mathcal{N} = (4, 4)$  gauge theories and DLCQ of ‘little string theories’,” *JHEP* **9910**, 030 (1999) [arXiv:hep-th/9909101].
- [9] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, “Linear dilatons, NS5-branes and holography,” *JHEP* **9810**, 004 (1998) [arXiv:hep-th/9808149].
- [10] C. G. Callan, J. A. Harvey and A. Strominger, “Supersymmetric string solitons,” arXiv:hep-th/9112030.
- [11] M. E. Peskin and D. V. Schroeder, “An Introduction To Quantum Field Theory,” Addison-Wesley, 1997, section 7.2.
- [12] A. Giveon, D. Kutasov and O. Pelc, “Holography for non-critical superstrings,” *JHEP* **9910**, 035 (1999) [arXiv:hep-th/9907178].
- [13] A. Giveon and D. Kutasov, “Little string theory in a double scaling limit,” *JHEP* **9910**, 034 (1999) [arXiv:hep-th/9909110].
- [14] A. Giveon and D. Kutasov, “Comments on double scaled little string theory,” *JHEP* **0001**, 023 (2000) [arXiv:hep-th/9911039].
- [15] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” *JHEP* **0002**, 020 (2000) [arXiv:hep-th/9912072].
- [16] M. Van Raamsdonk and N. Seiberg, “Comments on noncommutative perturbative dynamics,” *JHEP* **0003**, 035 (2000) [arXiv:hep-th/0002186].
- [17] P. Ginsparg and G. W. Moore, “Lectures On 2-D Gravity And 2-D String Theory,” arXiv:hep-th/9304011.
- [18] J. Teschner, “On structure constants and fusion rules in the  $SL(2, C)/SU(2)$  WZNW model,” *Nucl. Phys. B* **546**, 390 (1999) [arXiv:hep-th/9712256].

- [19] J. Teschner, “Operator product expansion and factorization in the  $H_3^+$  WZNW model,” Nucl. Phys. B **571**, 555 (2000) [arXiv:hep-th/9906215].
- [20] A. Giveon and D. Kutasov, “Notes on AdS(3),” Nucl. Phys. B **621**, 303 (2002) [arXiv:hep-th/0106004].
- [21] D. Kutasov and N. Seiberg, “More comments on string theory on AdS(3),” JHEP **9904**, 008 (1999) [arXiv:hep-th/9903219].
- [22] M. Bershadsky and D. Kutasov, “Comment on gauged WZW theory,” Phys. Lett. B **266**, 345 (1991).
- [23] R. Dijkgraaf, H. Verlinde and E. Verlinde, “String propagation in a black hole geometry,” Nucl. Phys. B **371**, 269 (1992).
- [24] J. M. Maldacena and H. Ooguri, “Strings in AdS(3) and the SL(2,R) WZW model. III: Correlation functions,” Phys. Rev. D **65**, 106006 (2002) [arXiv:hep-th/0111180].
- [25] P. Di Francesco and D. Kutasov, “World sheet and space-time physics in two-dimensional (Super)string theory,” Nucl. Phys. B **375**, 119 (1992) [arXiv:hep-th/9109005].
- [26] V. Fateev, A. B. Zamolodchikov and Al. B. Zamolodchikov, unpublished.
- [27] V. Kazakov, I. K. Kostov and D. Kutasov, “A matrix model for the two-dimensional black hole,” Nucl. Phys. B **622**, 141 (2002) [arXiv:hep-th/0101011].
- [28] G. Giribet and C. Nunez, “Correlators in AdS(3) string theory,” JHEP **0106**, 010 (2001) [arXiv:hep-th/0105200].
- [29] J. M. Maldacena and H. Ooguri, “Strings in AdS(3) and SL(2,R) WZW model. I,” J. Math. Phys. **42**, 2929 (2001) [arXiv:hep-th/0001053].
- [30] A. Parnachev and D. A. Sahakyan, “Some remarks on D-branes in AdS(3),” JHEP **0110**, 022 (2001) [arXiv:hep-th/0109150].
- [31] S. P. de Alwis, J. Polchinski and R. Schimmrigk, “Heterotic Strings With Tree Level Cosmological Constant,” Phys. Lett. B **218**, 449 (1989).
- [32] D. Kutasov and N. Seiberg, “Noncritical Superstrings,” Phys. Lett. B **251**, 67 (1990).
- [33] A. B. Zamolodchikov and V. A. Fateev, “Operator algebra and correlation functions in the two-dimensional Wess-Zumino  $SU(2) \times SU(2)$  chiral model,” Sov. J. Nucl. Phys. **43**, 657 (1986) [Yad. Fiz. **43**, 1031 (1986)].
- [34] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from  $N = 4$  super Yang Mills,” JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].
- [35] G. ’t Hooft, “A planar diagram theory for strong interactions,” Nucl. Phys. B **72**, 461 (1974).
- [36] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” JHEP **0006**, 008 (2000) [arXiv:hep-th/0003075].
- [37] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, “Giant gravitons in conformal field theory,” JHEP **0204**, 034 (2002) [arXiv:hep-th/0107119].

- [38] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual  $N = 4$  SYM theory,” *Adv. Theor. Math. Phys.* **5**, 809 (2002) [arXiv:hep-th/0111222]; S. Corley and S. Ramgoolam, “Finite factorization equations and sum rules for BPS correlators in  $N = 4$  SYM theory,” *Nucl. Phys. B* **641**, 131 (2002) [arXiv:hep-th/0205221].
- [39] O. Aharony, Y. E. Antebi, M. Berkooz and R. Fishman, “‘Holey sheets’: Pfaffians and subdeterminants as D-brane operators in large  $N$  gauge theories,” *JHEP* **0212**, 069 (2002) [arXiv:hep-th/0211152].
- [40] D. Berenstein, “Shape and holography: Studies of dual operators to giant gravitons,” *Nucl. Phys. B* **675**, 179 (2003) [arXiv:hep-th/0306090].
- [41] D. Berenstein, “A toy model for the AdS/CFT correspondence,” arXiv:hep-th/0403110.
- [42] P. C. Argyres and M. R. Douglas, “New phenomena in  $SU(3)$  supersymmetric gauge theory,” *Nucl. Phys. B* **448**, 93 (1995) [arXiv:hep-th/9505062].
- [43] P. C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, “New  $N = 2$  Superconformal Field Theories in Four Dimensions,” *Nucl. Phys. B* **461**, 71 (1996) [arXiv:hep-th/9511154].
- [44] T. Eguchi, K. Hori, K. Ito and S. K. Yang, “Study of  $N = 2$  Superconformal Field Theories in 4 Dimensions,” *Nucl. Phys. B* **471**, 430 (1996) [arXiv:hep-th/9603002].
- [45] I. R. Klebanov, “String theory in two-dimensions,” arXiv:hep-th/9108019.
- [46] J. Polchinski, “What is string theory?,” arXiv:hep-th/9411028.
- [47] J. Teschner, “Liouville theory revisited,” *Class. Quant. Grav.* **18**, R153 (2001) [arXiv:hep-th/0104158].
- [48] P. Di Francesco and D. Kutasov, “Correlation functions in 2-D string theory,” *Phys. Lett. B* **261**, 385 (1991).
- [49] J. M. Maldacena and A. Strominger, “Semiclassical decay of near-extremal five-branes,” *JHEP* **9712**, 008 (1997) [arXiv:hep-th/9710014].
- [50] O. Aharony and T. Banks, “Note on the quantum mechanics of M theory,” *JHEP* **9903**, 016 (1999) [arXiv:hep-th/9812237].
- [51] T. Harmark and N. A. Obers, “Hagedorn behaviour of little string theory from string corrections to NS5-branes,” *Phys. Lett. B* **485**, 285 (2000) [arXiv:hep-th/0005021].
- [52] M. Berkooz and M. Rozali, “Near Hagedorn dynamics of NS fivebranes, or a new universality class of coiled strings,” *JHEP* **0005**, 040 (2000) [arXiv:hep-th/0005047].
- [53] D. Kutasov and D. A. Sahakyan, “Comments on the thermodynamics of little string theory,” *JHEP* **0102**, 021 (2001) [arXiv:hep-th/0012258].
- [54] M. Rangamani, “Little string thermodynamics,” *JHEP* **0106**, 042 (2001) [arXiv:hep-th/0104125].
- [55] A. Buchel, “On the thermodynamic instability of LST,” arXiv:hep-th/0107102.
- [56] K. Narayan and M. Rangamani, “Hot little string correlators: A view from supergravity,” *JHEP* **0108**, 054 (2001) [arXiv:hep-th/0107111].

- [57] P. A. DeBoer and M. Rozali, “Thermal correlators in little string theory,” *Phys. Rev. D* **67**, 086009 (2003) [arXiv:hep-th/0301059].
- [58] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2**, 505 (1998) [arXiv:hep-th/9803131].
- [59] J. J. Atick and E. Witten, “The Hagedorn transition and the number of degrees of freedom of string theory,” *Nucl. Phys. B* **310**, 291 (1988).
- [60] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” *Phys. Rev. D* **58**, 046004 (1998) [arXiv:hep-th/9802042].
- [61] S. Sethi, “The matrix formulation of type IIB five-branes,” *Nucl. Phys. B* **523**, 158 (1998) [arXiv:hep-th/9710005].
- [62] O. J. Ganor and S. Sethi, “New perspectives on Yang-Mills theories with sixteen supersymmetries,” *JHEP* **9801**, 007 (1998) [arXiv:hep-th/9712071].
- [63] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” *Phys. Rev. D* **55**, 5112 (1997) [arXiv:hep-th/9610043].
- [64] L. Motl, “Proposals on nonperturbative superstring interactions,” arXiv:hep-th/9701025.
- [65] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” *Nucl. Phys. B* **500**, 43 (1997) [arXiv:hep-th/9703030].
- [66] I. Antoniadis, S. Dimopoulos and A. Giveon, “Little string theory at a TeV,” *JHEP* **0105**, 055 (2001) [arXiv:hep-th/0103033].
- [67] J. McGreevy and H. Verlinde, “Strings from tachyons: The  $c = 1$  matrix reloaded,” *JHEP* **0312**, 054 (2003) [arXiv:hep-th/0304224].
- [68] I. R. Klebanov, J. Maldacena and N. Seiberg, “D-brane decay in two-dimensional string theory,” *JHEP* **0307**, 045 (2003) [arXiv:hep-th/0305159].
- [69] T. Takayanagi and N. Toumbas, “A matrix model dual of type 0B string theory in two dimensions,” *JHEP* **0307**, 064 (2003) [arXiv:hep-th/0307083].
- [70] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, “A new hat for the  $c = 1$  matrix model,” arXiv:hep-th/0307195.
- [71] J. McGreevy, S. Murthy and H. Verlinde, “Two-dimensional superstrings and the supersymmetric matrix model,” arXiv:hep-th/0308105.
- [72] A. Giveon, A. Konechny, A. Pakman and A. Sever, “Type 0 strings in a 2-d black hole,” *JHEP* **0310**, 025 (2003) [arXiv:hep-th/0309056].
- [73] N. Arkani-Hamed, A. G. Cohen, D. B. Kaplan, A. Karch and L. Motl, “Deconstructing  $(2, 0)$  and little string theories,” *JHEP* **0301**, 083 (2003) [arXiv:hep-th/0110146].
- [74] N. Dorey, “S-duality, deconstruction and confinement for a marginal deformation of  $N = 4$  SUSY Yang-Mills,” arXiv:hep-th/0310117.